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A Stochastic Model for Weighted Loai Distribution with Properties and Its Applications

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ABSTRACT
Research and its application in the field of health are one of the most important applications of statistical analysis. In order to determine the best model for fitting the survival data, cancer research typically requires extra statistical analyses. Such data are well-fitted by existing classical distributions, but there has recently been an increase in interest in creating more flexible distributions by adding a few additional parameters to the fundamental model. A three-parameter continuous distribution has been introduced in this paper. A weighted model is used to produce the proposed distribution as a weighted Loai distribution. This distribution is a generalization of the base line distribution, which is the Loai distribution. It has been demonstrated to acquire several characteristics of the proposed distribution, such as the density function, distribution function, survival, hazard rate function, moments, moment generating function, entropies, order statistics, Bonferroni, and Lorenz curves. To estimate the model parameters based on the data, the maximum likelihood method is used. Through the use of several simulation studies, the effectiveness of these estimators is evaluated. Four data sets that indicate the survival of various cancer patients are used to empirically illustrate the potential importance and applications of the proposed distribution. The analysis's findings showed that the "Weighted Loai Distribution (WLD)" would perform well in practice when compared to some of the other distributions.
Keywords: *Weighted Model, Reliability Analysis, Statistical Properties, Entropies, Maximum Likelihood Estimation. Model Selection technic AIC, BIC, AICC.*

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I. INTRODUCTION

In their applied research, medical scientists are mostly concerned with investigating the survival of cancer patients. The analysis of time-to-event data, also known as survival of reliability data, has made substantial use of numerous statistical distributions in a variety of application sectors, including the medical industry. In order to locate and select the right model that effectively determines and estimates the survival data and produces trustworthy results and valid inferences, these investigations frequently need specific statistical analysis and modification. The concepts of size biased sampling and weighted distribution pertaining to observational studies and surveys of research related to forestry, ecology, bio-medicine, reliability, and several other areas have been widely studied in the literature. Adding extra parameter to an existing distribution brings the classical distribution in a more flexible situation and the distribution becomes useful for data analysis purpose. As a result, weighted distributions Aries naturally generated from a stochastic process and are recorded with some weighted function. When the weight function depends on the length of the unit of, interest, the resulting distribution is called weighted. Rao identified various situations that can be modeled by weighted distribution. An investigator records an observation by nature according to a certain stochastic model. The observation will not have the original distribution unless every observation is given an equal chance of being recorded. Suppose that the original observation X has $f(x)$ as the probability density function pdf (which may be probability density when X is continuous) and that the probability of recording the observation x is $0 \leq w(x) \leq 1$, then the pdf X_w, the observation is

$$f_w(x) = \frac{w(x)f(x)}{\omega}, x > 0$$

Where ω is the normalizing factor obtained to make the total probability equal to unity. With an arbitrary non-negative weight function $w(x)$ which may exceed unity, where $w(x) = x$ or $x^{\alpha}, \alpha > 0$ when $\alpha = 1$, are he called distributions with arbitrary $w(x)$ of is a special case. The weighted distribution with $w(x) = x$ is called the (weighted) or sized biased distribution. When the probability of observing a positive-valued random variable is proportional to the value of the variable the resultant is weighted distribution.

Weighted distribution was firstly introduced by Fisher (1934)[9] developed a new concept of distribution the weighted distribution, to model the ascertainment bias. The concept of weighted functions was first introduced by Rao (1965)[31] on discrete distributions arising out of ascertainment, then identified various situations that can be modelled by weighted distributions. A sampling plan that gives unequal probabilities to the various units by Patil and Rao (1977)[26] a weighted distributions a survey of their applications. Patil and Rao (1978)[27] weighted distributions and size-biased sampling with application to wildlife populations and human families. We study the properties of the weighted distributions in comparison with those of the original distributions for positive-valued random variables. Such random variables and distributions arise naturally in life testing, reliability, and economics. Blumental (1971)[7] proportional sampling in life length studies. Cox (1969)[6] some sampling problems in technology. Schaeffer (1972)[32] size-biased sampling. Mahfoud and Patil (1982)[22] on weighted distributions. Gupta (1986)[13] relations for reliability measures under length-biased sampling. Kochar and Gupta (1987)[19] some result on weighted distributions for positive-valued random variables, the weighted distributions have compared with the original distributions with the partial orderings of probability distributions. Also find out finally moments of the weighted distribution have been obtained. Oluyede (1999)[25] on inequalities and selection of experiments for weighted distribution occurs naturally for some

sampling plans in reliability, and survival analysis. Also, weighted distributions are proved for monotone hazard functions and mean residual life functions. Finally, entropy measures are also investigated. Blumenthal (1967)[4] and Scheaffer (1972)[4] the sampling mechanism selects units with probability proportional to some measure of the unit, the relating distribution is called a size biased. Size biased and weighted distributions have been used in etiological studies Simon (1980). Cnaan (1985)[7] on survival models with two phases and weighted sampling.

Recently, different authors have reviewed and studied the various probability weighted models illustrated their applications in different fields. Hashempour and Alizadeh (2023)[15] presented A New Weighted Half-Logistic Distribution: Properties, Applications and Different Method of Estimations. Ganaie and Rajagopalan (2023)[14] The Weighted Power Quasi Lindley Distribution with Properties and Applications of Life-time Data. Helal et al. (2022)[17] Statistical Properties of Weighted Shanker Distribution. Mohiuddin, et al. (2022)[23] discussed the Weighted Amarendra distribution: Properties and Applications to model real-life data. Chesneau, et al. (2022)[5] obtained the on a modified weighted exponential distribution with applications. Eyob, et al. (2019)[10] the author discussed by Weighted quasi-Akash distribution: properties and applications. Elangovan and Anthony (2019)[11] discussed Weighted OM distribution with properties and Applications to Survival Times. Atikankul, et al. (2020)[1] have been discussed the length-biased weighted Lindley distribution with applications. Hassan, et al. (2019)[16] A new generalization of Pranav distribution using weighted technique. Shanker, and Shukla (2017)[33] A quasi Shanker distribution and its applications. Alqallaf, et al. (2015)[2] Weighted exponential distribution: Different methods of estimations.

In this research, we adopt the idea of a propose a new three parameter distribution. The proposed distribution such that, weighted Loai distribution. The Loai distribution introduced by Loai Alzoui et. al, (2022)[21] is a newly proposed three parameter lifetime model for various medical science applications. The proposed distribution Weighted Loai distribution (WLD) shows its flexibility and superiority to fit some real lifetime data sets compared to some competing distributions.

The present paper is organized, as a In Section 2, we derived the probability density function (pdf) and cumulative distribution function (cdf) of the weighted Loai distribution. In Section 3, we discussed in reliability analysis of the weighted Loai distribution. In Section 4, we have some statistical properties, including moment-generating function, r^{th} moment, mean, variance, coefficient of variance, and hormonic mean. In Section 5, we derived mean deviation. In Section 6, mean deviation from the median. In Section 6, we explore the distribution of order statistics and, the quantile function. In Section 7, likelihood ratio test. In Section 8, we present Bonferroni and Lorenz curves and the Gini index. Section 9, provides the stochastic ordering of the distribution. Entropies are derived In Section 10. Maximum likelihood estimates and Fisher's information matrix, we have derived Section 11. Finally, different applications of the weighted Loai distribution to complete and censored datasets are presented Section 12. All computations throughout this paper were performed using the statistical programming language R. The Conclusion is presented in Section 13.

II. WEIGHTED LOAI DISTRIBUTION

In this section, we define the probability density function (pdf) and cumulative distribution function (cdf) of the weighted Loai distribution.
A new two parameter life time distribution name as Loai distribution. The probability density function (pdf) of the Loai distribution is given by

$$f(x)=\frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\qquad ;x>0,\alpha>0,\theta>0 \qquad (1)$$

And cumulative distribution function of the Loai distribution is given by

$$F(x)=1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x\right]e^{-\theta x}\qquad ;x>0,\alpha>0,\theta>0 \qquad (2)$$

We have considered a random variable x with a probability density function $f(x)$. Let $w(x)$ be a non-negative weight function. Denote a new probability density function.

$$f_w(x)=\frac{w(x)f(x)}{E[w(x)]}\ ;x>0$$

Where $w(x)$ be the non-negative weight function and $E[w(x)]=\int w(x)f(x)dx<\infty$
and corresponding random variable by X_w, which is called the weighted random variable corresponding to x. When $w(x)=x^c$, $c>0$, we say that X_w size-biased of order c. such a selection procedure is called size-biased sampling of order c. when $c=1,2$, X_w is simply called size-biased (or weighted) and has a probability density function

$$f_w(x)=\frac{xf(x)}{E(X)}\quad ;\ x>0$$

Gupta (1986)[14] has obtained some relations between the reliability measures of the original distribution and those of the weighted distribution.
The weighted distribution is obtained by applying the weighted function as $w(x)=x^c$, in weights we use $w(x)=x^c$ to the Loai distribution in order to obtain the weighted Loai distribution. The probability density function of the weighted Loai distribution given by

$$f_w(x;\alpha,\theta)=\frac{x^c\,f(x;\theta)}{E(X^c)}\qquad ;x>0,\theta>0,>0 \qquad (3)$$

Where,

$$E(X)=\int_0^\infty x^c\ f(x;\theta)\,dx$$

After simplification we get, gamma function is given by

$$\Gamma(Z)=\int_0^\infty t^{z-1}\ e^{-t}dt$$

$$E(X)=\left(\frac{\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}}{\theta^c(\alpha+1)}\right) \qquad (4)$$

Substituting the value of equation (1) and (4) in equation (3), we get the probability density function (pdf) of weighted Loai distribution.

$$f_w(x;\alpha,\theta,c)=\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\,x^c\left(\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right)e^{-\theta x} \qquad (5)$$

After simplification, using a lower incomplete gamma function is given by

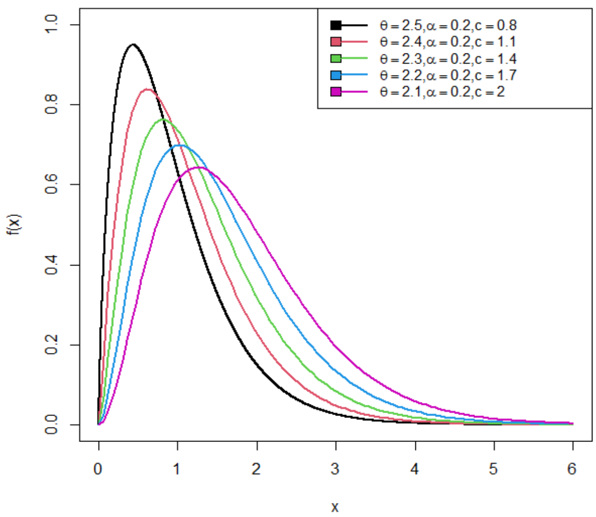
$$\gamma\big((z+1),\theta x\big)=\int_0^{\theta x} t^{(z+1)-1}e^{-t}dt$$

We will get Cumulative distribution function of weighted Loai distribution is given by

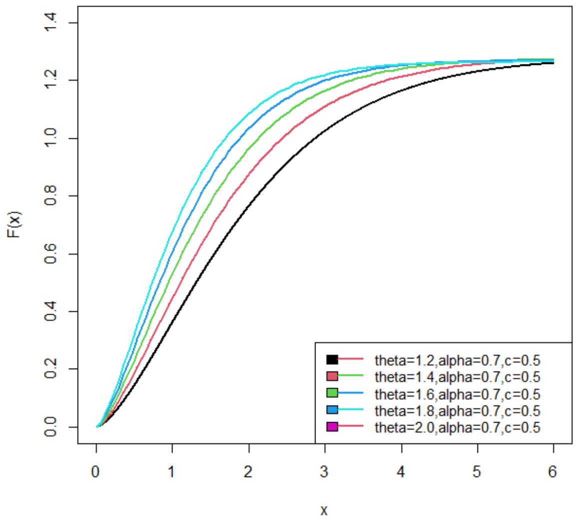
$$F_w(x;\alpha,\theta,c)=\int_0^x f_w(x;\alpha,\theta,c)\ dx$$

After simplification of equation (6), we obtain the cumulative distribution function of weighted Loai distribution

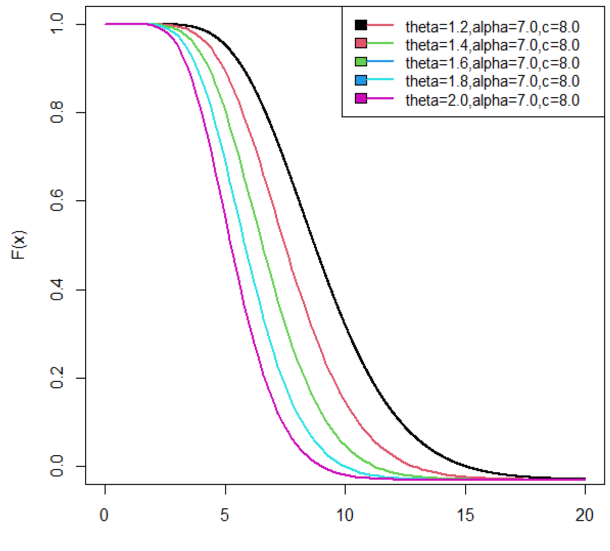
$$F_w(x;\alpha,\theta,c)=\frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x)+\frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x)+\gamma(c+2,\theta x)\big)\right)}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)} \qquad (7)$$



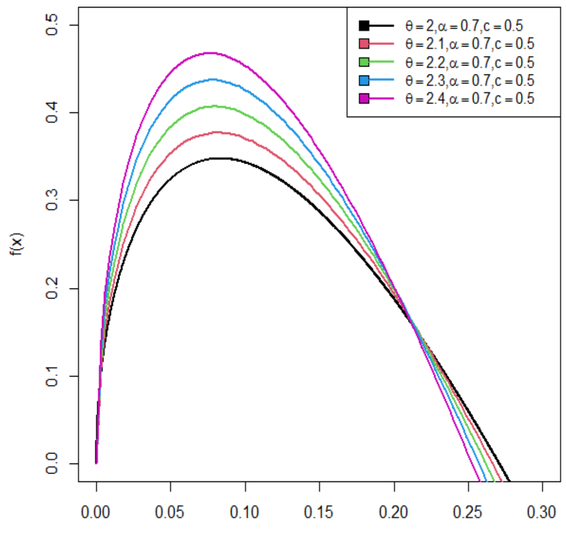
Figures.1:Pdf plot of Weighted Loai distribution



Figures.2 Cdf plot of Weighted Loai distribution



Figures.3 survival function of Weighted Loai distribution



Figures.4:Hazard function of Weighted Loai distribution

III. RELIABILITY ANALYSIS

In this section, we will discuss the reliability function, hazard function, reverse hazard function, cumulative hazard function, Odds rate, Mills ratio and, Mean Residual function for the proposed weighted Loai distribution.

3.1 Reliability function

The reliability function is also known as survival function. It can be computed as complement of the cumulative distribution function. The survival function of weighted Loai distribution is given by

$$S(x) = 1 - F_w(x; \alpha, \theta, c)$$

$$S(x) = 1 - \frac{\left(\frac{\alpha}{2}\gamma(c+3, \theta x) + \frac{1}{\theta+1}\left(\theta\, \gamma(c+1, \theta x) + \gamma(c+2, \theta x)\right)\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}$$

3.2 Hazard function

The Hazard function is also known as hazard rate, instantaneous failure rate or force mortality of weighted Loai distribution is given by

$$h(x) = \frac{f_w(x;\alpha,\theta,c)}{1 - F_w(x;\alpha,\theta,c)}$$

$$h(x) = \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right) - \left(\frac{\alpha}{2}\gamma(c+3,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x) + \gamma(c+2,\theta x)\big)\right)} x^c \times \left[\frac{1}{2}\,\alpha\theta\,x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x}$$

3.3 Revers hazard rate

Reverse hazard function of weighted Loai distribution is given by

$$h_r(x) = \frac{f_w(x;\alpha,\theta,c)}{F_w(x;\alpha,\theta,c)}$$

$$h_r(x) = \frac{\theta^{c+2}}{\left(\frac{\alpha}{2}\gamma(c+3,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x) + \gamma(c+2,\theta x)\big)\right)} \times\ x^c \left[\frac{1}{2}\,\alpha\theta\,x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x}$$

3.4 Odds rate function

Odds Rate function of weighted Loai distribution is given by

$$O(x) = \frac{F_w(x;\alpha,\theta,c)}{1 - F_w(x;\alpha,\theta,c)}$$

$$O(x) = \frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x) + \gamma(c+2,\theta x)\big)\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right) - \left(\frac{\alpha}{2}\gamma(c+3,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x) + \gamma(c+2,\theta x)\big)\right)}$$

3.5 Cumulative hazard function

Cumulative hazard function of weighted Loai distribution is given by

$$H(x) = -\ln\big(1 - F_w(x;\alpha,\theta,c)\big)$$

$$H(x) = \ln\left(\frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x) + \gamma(c+2,\theta x)\big)\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} - 1\right)$$

3.6 Mills Ratio

$$\text{Mills Ratio} = \frac{1}{h_r(x)}$$

$$\text{Mills Ratio } = \frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x) + \gamma(c+2,\theta x)\big)\right)}{\theta^{c+2}\ \ x^c\ \ \left[\frac{1}{2}\,\alpha\theta\,x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x}}$$

3.7 Mean Residual function

Mean Residual function of weighted Loai distribution is given by

$$M(x) = \frac{1}{S(x)}\int_x^\infty t\,f(t)dt - x$$

$$M(x) = \frac{1}{1 - \frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x) + \gamma(c+2,\theta x)\big)\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}} \times \int_x^\infty t\,\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}\,t^c\ \left[\frac{1}{2}\,\alpha\theta\,t^2 + \frac{1}{\theta+1}(1+t)\right] e^{-\theta t} - x$$

$$M(x)=\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)-\left(\frac{\alpha}{2}\gamma(c+3,\theta x)+\frac{1}{\theta+1}\left(\theta\,\gamma(c+1,\theta x)+\gamma(c+2,\theta x)\right)\right)}\times\int_x^\infty t^{c+1}\left[\frac{1}{2}\alpha\theta\,t^2+\frac{1}{\theta+1}(1+t)\right]e^{-\theta t}-x$$

Put $\theta t=x,\quad t=\frac{x}{\theta},\quad dt=\frac{1}{\theta}\,dx$

When $x\to 0, t\to 0$ and $x\to x, t\to\theta x$

After solving the integral, we get

Upper incomplete gamma function

$$\Gamma(s,x)=\int_x^\infty t^{s-1}\,e^{-t}\,dt$$

$$M(x)=\frac{\left(\frac{\alpha}{2}\Gamma(c+4,\theta x)+\frac{1}{\theta+1}\left(\theta\Gamma(c+2,\theta x)+\Gamma(c+3,\theta x)\right)\right)}{\theta\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)-\left(\frac{\alpha}{2}\gamma(c+3,\theta x)+\frac{1}{\theta+1}\left(\theta\,\gamma(c+1,\theta x)+\gamma(c+2,\theta x)\right)\right)}$$

IV. STATISTICAL PROPERTIES

In this section, we derived the structural properties, the moment generating function, Characteristic function and r[th] moment for the weighted Loai distribution random variable are derived. Also, the mean, variance, standard deviation, hormonic mean is investigated.

4.1 Moments

Let X_l denoted the random variable following weighted Loai distribution then r^{th} order moments $E(X^r)$ is obtained as

$$E(X^r)=\mu_r'=\int_0^\infty x^r\ f_w(x;\alpha,\theta,c)\,dx$$

$$\mu_r'=\int_0^\infty x^r\ \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\ x^c\ \left(\frac{1}{2}\ \alpha\theta\ x^2+\frac{1}{\theta+1}(1+x)\right)e^{-\theta x}dx$$

$$\mu_r'=\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\int_0^\infty\ x^{c+r}\ \left(\frac{1}{2}\ \alpha\theta\ x^2+\frac{1}{\theta+1}(1+x)\right)e^{-\theta x}dx$$

$$\mu_r'=\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\left(\frac{1}{2}\ \alpha\theta\int_0^\infty\ x^{c+r+2}\ e^{-\theta x}\ dx\right.$$
$$\left.+\frac{1}{\theta+1}\left(\int_0^\infty x^{r+c}\ e^{-\theta x}\ dx+\int_0^\infty x^{c+r+1}\ e^{-\theta x}\ dx\right)\right)$$

$$\mu_r'=\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\left(\frac{1}{2}\ \alpha\theta\ \frac{\Gamma(c+r+3)}{\theta^{c+r+3}}+\frac{1}{\theta+1}\left(\frac{\Gamma(c+r+1)}{\theta^{c+r+1}}+\frac{\Gamma(c+r+2)}{\theta^{c+r+2}}\right)\right)$$

$$\mu_r'=\frac{1}{\theta^r\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\left(\frac{\alpha\Gamma(c+r+3)}{2}\ +\frac{\left(\theta\Gamma(c+r+1)+\Gamma(c+r+2)\right)}{\theta+1}\right)$$

$$\mu_r'=\frac{\left(\frac{\alpha(c+r+2)!}{2}\ +\frac{(\theta(c+r)!+(c+r+1)!)}{\theta+1}\right)}{\theta^r\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\qquad(8)$$

Putting $r=1,2,3,4$ in equation (8), the mean of weighted Loai distribution is obtained as

$$\mu_1'=\frac{\left(\frac{\alpha(c+3)!}{2}\ +\frac{(\theta(c+1)!+(c+2)!)}{\theta+1}\right)}{\theta\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}$$

$$\mu_2'=\frac{\left(\frac{\alpha(c+4)!}{2}\ +\frac{(\theta(c+2)!+(c+3)!)}{\theta+1}\right)}{\theta^2\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}$$

$$\mu_3' = \frac{\left(\frac{\alpha(c+5)!}{2} + \frac{(\theta(c+3)! + (c+4)!)}{\theta+1}\right)}{\theta^3\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}$$

$$\mu_4' = \frac{\left(\frac{\alpha(c+6)!}{2} + \frac{(\theta(c+4)! + (c+5)!)}{\theta+1}\right)}{\theta^4\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}$$

Variance $= \mu_2' - (\mu_1')^2$

$$\sigma^2 = \frac{\left(\frac{\alpha(c+4)!}{2} + \frac{(\theta(c+2)! + (c+3)!)}{\theta+1}\right)}{\theta^2\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} - \left(\frac{\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)}{\theta\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}\right)^2$$

$$\sigma^2 = \frac{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)\left(\frac{\alpha(c+4)!}{2} + \frac{(\theta(c+2)! + (c+3)!)}{\theta+1}\right) - \left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)^2}{\theta^2\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)^2}$$

Standard Deviation

$$S.D(\sigma) = \sqrt{\frac{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)\left(\frac{\alpha(c+4)!}{2} + \frac{(\theta(c+2)! + (c+3)!)}{\theta+1}\right) - \left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)^2}{\theta\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}}$$

4.2 Hormonic Mean

The Hormonic mean of the weighted Loai distribution is defined as

$$H.M = E\left[\frac{1}{x}\right] = \int_0^\infty \frac{1}{x}\, f_w(x;\theta,\alpha,c)\, dx$$

$$H.M = \int_0^\infty \frac{1}{x} \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}\, x^c \left(\frac{1}{2}\alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right) e^{-\theta x}\, dx$$

$$H.M = \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \int_0^\infty x^{c-1} \left(\frac{1}{2}\alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right) e^{-\theta x}\, dx$$

$$H.M = \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \int_0^\infty \left(\frac{1}{2}\alpha\theta\, x^{c+1} + \frac{1}{\theta+1}(x^{c-1} + x^c)\right) e^{-\theta x}\, dx$$

$$H.M = \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \left(\frac{1}{2}\alpha\theta \int_0^\infty x^{c+1} e^{-\theta x}\, dx + \frac{1}{\theta+1}\left(\int_0^\infty x^{c-1} e^{-\theta x}\, dx + \int_0^\infty x^c e^{-\theta x}\, dx\right)\right)$$

$$H.M = \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \left(\frac{1}{2}\alpha\theta \frac{(c+1)!}{\theta^{c+2}} + \frac{1}{\theta+1}\left(\frac{(c-1)!}{\theta^c} + \frac{c!}{\theta^{c+1}}\right)\right)$$

$$H.M = \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \frac{1}{\theta^{c+1}} \left(\frac{\alpha(c+1)!}{2} + \frac{\theta(c-1)! + c!}{\theta+1}\right)$$

$$H.M = \frac{\theta\left(\frac{\alpha(c+1)!}{2} + \frac{\theta(c-1)! + c!}{\theta+1}\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}$$

4.3 Moment Generating function and Characteristic function

Let X_l follows weighted Loai distribution then the moment generating function (mgf) of X is obtained as

$$M_{X_w}(t) = E(e^{tx}) = \int_0^\infty e^{tx}\, f_W(x;\alpha,\theta,c)\, dx$$

Using Taylor's series

$$M_{X_w}(t) = \int_0^\infty \left[1 + tx + \frac{(tx)^2}{2!} + \cdots\right] f_w(x;\alpha,\theta,c)\, dx$$

$$M_{X_w}(t) = \int_0^\infty \sum_{j=0}^{\infty} \frac{t^j}{j!}\, x^j\;\, f_w(x;\alpha,\theta,c)\, dx$$

$$M_{X_w}(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^\infty x^j\;\, f_w(x;\alpha,\theta,c)\, dx$$

$$M_{X_w}(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\quad E\left(X_l^j\right)$$

$$M_{X_w}(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{\left(\frac{\alpha(c+j+2)!}{2} \;+ \frac{(\theta(c+j)! + (c+j+1)!)}{\theta+1}\right)}{\theta^r\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}$$

$$M_{X_w}(t) = \frac{1}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \sum_{j=0}^{\infty} \frac{t^j}{j!\,\theta^j} \times \left(\frac{\alpha(c+j+2)!}{2} \;+ \frac{(\theta(c+j)! + (c+j+1)!)}{\theta+1}\right)$$

Similarly, we can get the characteristic function of weighted Loai distribution can be obtained as

$$\phi_{X_w}(t) = M_{X_w}(it)$$

$$\phi_{X_w}(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!}\;\; \mu_j'$$

$$\phi_{X_w}(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!}\; \frac{\left(\frac{\alpha(c+j+2)!}{2} \;+ \frac{(\theta(c+j)! + (c+j+1)!)}{\theta+1}\right)}{\theta^r\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}$$

$$\phi_{X_w}(t) = \frac{1}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \sum_{j=0}^{\infty} \frac{(it)^j}{j!\,\theta^j} \times \left(\frac{\alpha(c+j+2)!}{2} \;+ \frac{(\theta(c+j)! + (c+j+1)!)}{\theta+1}\right)$$

V. MEAN DEVIACTION

Let X be a random variable from weighted Loai distribution with mean μ. Then the deviation from mean is defined as

$$D(\mu) = E(|X-\mu|)$$

$$D(\mu) = \int_0^\infty |X-\mu|\;\, f_W(x;\alpha,\theta,c)dx$$

$$D(\mu) = \int_0^\mu (\mu - x) f_W(x;\alpha,\theta,c)dx + \int_\mu^\infty (x-\mu) f_W(x;\alpha,\theta,c)dx$$

$$D(\mu) = \mu \int_0^\mu f_W(x;\alpha,\theta,c)dx - \int_0^\mu x\, f_W(x;\alpha,\theta,c)dx + \int_\mu^\infty x\, f_W(x;\alpha,\theta,c)dx - \int_\mu^\infty \mu\, f_W(x;\alpha,\theta,c)dx$$

$$D(\mu) = \mu F(\mu) - \int_0^\mu x\, f_W(x;\alpha,\theta,c)dx - \mu[1 - F(\mu)] + \int_\mu^\infty x\, f_W(x;\alpha,\theta,c)dx$$

$$D(\mu) = 2\mu F(\mu) - 2\int_0^\mu x\, f_W(x;\alpha,\theta,c)dx$$

Now,

$$\int_0^\mu x\, f_W(x;\alpha,\theta,c)dx = \int_0^\mu x\; \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}\; x^c\; \left(\frac{1}{2}\;\alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right) e^{-\theta x} dx$$

$$= \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \int_0^\mu \;\; x^{c+1}\; \left(\frac{1}{2}\;\alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right) e^{-\theta x} dx$$

Put $\theta x = t, \quad x = \frac{t}{\theta}, \quad dx = \frac{1}{\theta} dt$

When $\mu \to 0, t \to 0\ and\ \mu \to \mu, t \to \theta\mu$

$$= \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \int_0^{\theta\mu} \frac{t^{c+1}}{\theta^{c+1}} \left(\frac{1}{2}\alpha\theta \frac{t^2}{\theta^2} + \frac{1}{\theta+1}\left(1+\frac{t}{\theta}\right)\right) e^{-t} \frac{1}{\theta} dx$$

After solving the integral, we get

$$D(\mu) = \frac{2\mu\left(\frac{\alpha}{2}\gamma(c+3,\theta\mu) + \frac{1}{\theta+1}\left(\theta\,\gamma(c+1,\theta\mu) + \gamma(c+2,\theta\mu)\right)\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} - \frac{2\left(\frac{\alpha}{2}\gamma(c+4,\theta\mu) + \frac{1}{\theta+1}\left(\theta\,\gamma(c+2,\theta\mu) + \gamma(c+3,\theta\mu)\right)\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}$$

VI. MEAN DIVIATION FROM MEDIAN

Let X be a random variable from weighted Loai distribution with median M. Then the mean deviation from median is defined as

$$D(M) = E(|X - M|)$$

$$D(M) = \int_0^\infty |X - M|\ f_W(x;\alpha,\theta,c)dx$$

$$D(M) = \int_0^M (M - x) f_W(x;\alpha,\theta,c)dx + \int_M^\infty (x - M) f_W(x;\alpha,\theta,c)dx$$

$$D(M) = MF(M) - \int_0^M x\, f_W(x;\alpha,\theta,c)dx - M[1 - F(M)] + \int_M^\infty x\, f_W(x;\alpha,\theta,c)dx$$

$$D(M) = \mu - 2\int_0^M x\, f_W(x;\alpha,\theta,c)dx$$

Now,

$$\int_0^M x\, f_W(x;\alpha,\theta,c)dx = \int_0^M x \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} x^c \left(\frac{1}{2}\alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right) e^{-\theta x} dx$$

$$= \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \int_0^M x^{c+1} \left(\frac{1}{2}\alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right) e^{-\theta x} dx$$

Put $\theta x = t, \quad x = \frac{t}{\theta}, \quad dx = \frac{1}{\theta} dt$

When $M \to 0, t \to 0\ and\ M \to M, t \to \theta M$

After solving the integral, we get

$$D(M) = \mu - \frac{2\left(\frac{\alpha}{2}\gamma(c+4,\theta M) + \frac{1}{\theta+1}\left(\theta\,\gamma(c+2,\theta M) + \gamma(c+3,\theta M)\right)\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}$$

VII. ORDER STATISTICS

In this section, we derived the distributions of order statistics from the weighted Loai distribution.

Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ be the order statistics of the random sample $X_1, X_2, X_3, \ldots, X_n$ selected from weighted Loai distribution. Then the probability density function of the k^{th} order statistics $X_{(k)}$ is defined as.

$$f_{X_{w(k)}}(x) = \frac{n!}{(r-1)!\,(n-k)!} f_X(x)\,[F_X(x)]^{k-1}\,[1 - F_X(x)]^{n-k} \qquad (9)$$

Inserting equation (5) and (7) in equation (9), the probability density function of k^{th} order statistics $X_{(k)}$ of the weighted Loai distribution is given by

$$f_{X_{(k)}}(x) = \frac{n!}{(n-1)!\,(n-k)!}\left(\frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x)+\frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x)+\gamma(c+2,\theta x)\big)\right)}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\right)^{k-1}$$

$$\times\left(1-\frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x)+\frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x)+\gamma(c+2,\theta x)\big)\right)}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\right)^{n-k}$$

$$\times\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\,x^c\left(\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right)e^{-\theta x}$$

The distribution of the minimum first order statistics $X_{(1)} = \min(X_1, X_2, X_3, \dots, X_n)$ and the largest order statistics $X_{(n)} = max(X_1, X_2, X_3, \dots, X_n)$ can be computed by replacing k in the previous equation by 9 and n, so we get.

$$f_{X_{(1)}}(x) = \frac{n\,\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\,x^c\left(\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right)e^{-\theta x}$$

$$\times\left(1-\frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x)+\frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x)+\gamma(c+2,\theta x)\big)\right)}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\right)^{n-1}$$

$$f_{X_{(n)}}(x) = \frac{n\ \ \theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\,x^c\left(\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right)e^{-\theta x}$$

$$\times\left(\frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x)+\frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x)+\gamma(c+2,\theta x)\big)\right)}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\right)^{n-1}$$

Quantile function

The quantile function of a distribution with cumulative distribution function $F_l(x;\alpha,\theta)$ is defined by $q = F_l(x_q;\alpha,\theta)$, where $0 < q < 1$. Thus, the quantile function of weighted Loai distribution is the real solution of the equation.

$$1-q = 1-\frac{\left(\frac{\alpha}{2}\gamma(c+3,\theta x)+\frac{1}{\theta+1}\big(\theta\,\gamma(c+1,\theta x)+\gamma(c+2,\theta x)\big)\right)}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}$$

VIII. LIKELIHOOD RATIO TEST

In this section, we derive the likelihood ratio test from the weighted Loai distribution.
Let $x_1, x_2, x_3, \dots, x_n$ be a random sample from the weighted Loai distribution.
To testing the hypothesis, we have the null and alternative hypothesis.

$H_0: f(x) = f(x,\alpha,\theta)$ $\quad$ *against* $\quad$ $H_1: f(x) = f_l(x;\alpha,\theta,c)$

In test whether the random sample of size n comes from the Loai distribution or weighted Loai distribution, the following test statistics is used.

$$\Delta = \frac{L_1}{L_2} = \prod_{i=1}^{n}\frac{f_l(x_i;\alpha,\theta)}{f(x_i,\theta)}$$

$$\Delta = \prod_{i=1}^{n}\left(\frac{\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\,x_i^c\left[\frac{1}{2}\,\alpha\theta\,x_i^2+\frac{1}{\theta+1}(1+x_i)\right]e^{-\theta x_i}}{\frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\,\alpha\theta\,x_i^2\ +\frac{1}{\theta+1}(1+x_i)\right]e^{-\theta x_i}}\right)$$

$$\Delta = \prod_{i=1}^{n} \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} x_i^c \left[\frac{1}{2}\, \alpha\theta\, x_i^2 + \frac{1}{\theta+1}(1+x_i)\right] e^{-\theta x_i} \times \frac{\alpha+1}{\theta^2 \left[\frac{1}{2}\, \alpha\theta\, x_i^2 + \frac{1}{\theta+1}(1+x_i)\right] e^{-\theta x_i}}$$

$$\Delta = \prod_{i=1}^{n} \left(\frac{\theta^c\, (\alpha+1)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} x_i^c \right)$$

$$\Delta = \frac{L_1}{L_2} \left(\frac{\theta^c\, (\alpha+1)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \right)^n \prod_{i=1}^{n} x_i^c$$

We have rejected the null hypothesis if

$$\Delta = \left(\frac{\theta^c\, (\alpha+1)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \right)^n \prod_{i=1}^{n} x_i^c > k$$

Equivalently, we also reject null hypothesis, where

$$\Delta^* = \prod_{i=1}^{n} x_i^c > k \left(\frac{\theta^c\, (\alpha+1)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i^c > k^* \; where \; k^* = k \left(\frac{\theta^c\, (\alpha+1)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \right)^n$$

for large sample size n, 2 $\log \Delta$ is distribution as chi-square variates with one degree of freedom. Thus, we rejected the null hypothesis, when the probability value is given by $p(\Delta^* > \alpha^*)$, where $\alpha^* = \prod_{i=1}^{n} x_i^c$ is less than level of significance and $\prod_{i=1}^{n} x_i^c$ is the observed value of the statistics Δ^*.

IX. BONFERRONI AND LORENZ CURVES AND GINI INDEX

In this section, we have derived the Bonferroni and Lorenz curves and Gini index from the weighted Loai distribution.

The Bonferroni and Lorenz curve is a powerful tool in the analysis of distributions and has applications in many fields, such as economies, insurance, income, reliability, and medicine. The Bonferroni and Lorenz cures for a X be the random variable of a unit and $f(x)$ be the probability density function of x. $f(x)dx$ will be represented by the probability that a unit selected at random is defined as

$$B(p) = \frac{1}{p\mu} \int_0^q x\, f_w(x; \alpha, \theta, c) dx$$

And

$$L(p) = \frac{1}{\mu} \int_0^q x\, f_w(x; \alpha, \theta, c) dx$$

Where, $q = F^{-1}(p); \quad q \epsilon [0,1]$

and $\mu = E(X)$

Hence the Bonferroni and Lorenz corves of our distribution are, given by

$$\mu = \frac{\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)}{\theta \left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}$$

$$B(p) = \frac{\theta \left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)}{p \left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)} \int_0^q x\, \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} x^c \times \left(\frac{1}{2}\, \alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right) e^{-\theta x}$$

$$B(p) = \frac{\theta^{c+3}}{p\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)} \int_0^q x^{c+1} \left[\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x}$$

Put $\theta x = t, \quad x = \frac{t}{\theta}, \quad dx = \frac{1}{\theta} dt$

When $x \to 0, t \to 0\ and\ x \to x, t \to \theta q$

$$B(p) = \frac{\theta^{c+3}}{p\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)} \times \int_0^{\theta q} \frac{t^{c+1}}{\theta^{c+1}} \left(\frac{1}{2}\alpha\theta \frac{t^2}{\theta^2} + \frac{1}{\theta+1}\left(1 + \frac{t}{\theta}\right)\right) e^{-t} \frac{1}{\theta} dt$$

$$B(p) = \frac{\theta^{c+3}}{p\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)} \times \int_0^{\theta q} \frac{t^{c+1}}{\theta^{c+1}} \left(\frac{1}{2}\alpha\theta \frac{t^2}{\theta^2} + \frac{1}{\theta+1}\left(\frac{\theta}{\theta} + \frac{t}{\theta}\right)\right) e^{-t} \frac{1}{\theta} dt$$

$$B(p) = \frac{\theta^{c+3}}{p\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)} \frac{1}{\theta^{c+3}} \times \int_0^{\theta q} t^{c+1} \left(\frac{1}{2}\alpha\theta \frac{t^2}{\theta^2} + \frac{1}{\theta+1}\left(\frac{\theta}{\theta} + \frac{t}{\theta}\right)\right) e^{-t} \frac{1}{\theta} dt$$

$$B(p) = \frac{1}{p\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)} \times \left(\frac{\alpha}{2}\int_0^{\theta q} t^{c+3} e^{-t} dt + \frac{1}{\theta+1}\left(\theta \int_0^{\theta q} t^{c+1} e^{-t} dt + \int_0^{\theta q} t^{c+2} e^{-t} dt\right)\right)$$

After solving the integral, we get

$$B(p) = \frac{\left(\frac{\alpha}{2}\gamma(c+4, \theta q) + \frac{1}{\theta+1}\left(\theta\gamma(c+2, \theta q) + \gamma(c+3, \theta q)\right)\right)}{p\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)}$$

Where,

$L(p) = pB(p)$

$$L(p) = \frac{\left(\frac{\alpha}{2}\gamma(c+4, \theta q) + \frac{1}{\theta+1}\left(\theta\gamma(c+2, \theta q) + \gamma(c+3, \theta q)\right)\right)}{\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)}$$

Gini Index

The information in the Lorenz Curve is often summarized in a single measure called the Gini index (proposed in a 1912 paper by the Italian statistician Corrado Gini. It is often used as a gauge of economic inequality, measuring income distribution. The Gini index is defined as

Therefore, the Gini index is for weighted Loai distribution

$$G = 1 - 2\int_0^1 L(p) dp$$

$$G = 1 - 2\int_0^1 \frac{\left(\frac{\alpha}{2}\gamma(c+4, \theta q) + \frac{1}{\theta+1}\left(\theta\gamma(c+2, \theta q) + \gamma(c+3, \theta q)\right)\right)}{\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)} dp$$

$$G = 1 - 2\frac{\left(\frac{\alpha}{2}\gamma(c+4, \theta q) + \frac{1}{\theta+1}\left(\theta\gamma(c+2, \theta q) + \gamma(c+3, \theta q)\right)\right)}{\left(\frac{\alpha(c+3)!}{2} + \frac{(\theta(c+1)! + (c+2)!)}{\theta+1}\right)}$$

X. STOCHASTIC ORDERING

Stochastic ordering is an important tool in finance and reliability to assess the comparative performance of the models. Let X and Y be two random variables with pdf, cdf, and reliability functions $f(x), f(y), F(x), F(y). S(x) = 1 - F(x) and\ F(y)$.

1- Likelihood ratio order ($X \leq_{LR} Y$) if $\frac{f_{X_l}(x)}{f_{Y_l}(x)}$ decreases in x

2- Stochastic order ($X \leq_{ST} Y$) if $F_{X_l}(x) \geq F_{Y_l}(x)$ for all x

3- Hazard rate order ($X \leq_{HR} Y$) if $h_{X_l}(x) \geq h_{Y_l}(x)$ for all x

4- Mean residual life order ($X \leq_{MRL} Y$) if $MRL_{X_l}(X) \leq MRL_{Y_w}(X)$ for all x

Show that weighted Loai distribution satisfies the strongest ordering (likelihood ratio ordering)

Assume that X and Y are two independent Random variables with probability distribution function$f_{l_x}(x; \alpha, \theta, c_1)$ and $f_{l_y}(x; \beta, \lambda, c_2)$. If $\alpha < \beta < c_1$ and $\theta < \lambda < c_2$, then

$$\Lambda = \frac{f_{l_x}(x; \alpha, \theta, c_1)}{f_{l_y}(x; \beta, \lambda, c_2)}$$

$$\Lambda = \frac{\frac{\theta^{c_1+2}}{\left(\frac{\alpha(c_1+2)!}{2} + \frac{(\theta c_1! + (c_1+1)!)}{\theta+1}\right)} x^{c_1} \left(\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right) e^{-\theta x}}{\frac{\lambda^{c_2+2}}{\left(\frac{\beta(c_2+2)!}{2} + \frac{(\theta c_2! + (c_2+1)!)}{\lambda+1}\right)} x^{c_2} \left(\frac{1}{2}\beta\lambda x^2 + \frac{1}{\lambda+1}(1+x)\right) e^{-\lambda x}}$$

$$\Lambda = \frac{\theta^{c_1+2}}{\left(\frac{\alpha(c_1+2)!}{2} + \frac{(\theta c_1! + (c_1+1)!)}{\theta+1}\right)} x^{c_1} \left(\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right) e^{-\theta x} \times \frac{\left(\frac{\beta(c_2+2)!}{2} + \frac{(\theta c_2! + (c_2+1)!)}{\lambda+1}\right)}{\lambda^{c_2+2} x^{c_2} \left(\frac{1}{2}\beta\lambda x^2 + \frac{1}{\lambda+1}(1+x)\right) e^{-\lambda x}}$$

$$= \frac{\left(\theta^{c_1+2}\left(\frac{\beta(c_2+2)!}{2} + \frac{(\theta c_2! + (c_2+1)!)}{\lambda+1}\right)\right)}{\left(\lambda^{c_2+2}\left(\frac{\alpha(c_1+2)!}{2} + \frac{(\theta c_1! + (c_1+1)!)}{\theta+1}\right)\right)} \times \frac{\left(x^{c_1}\left(\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right)\right)}{\left(x^{c_2}\left(\frac{1}{2}\beta\lambda x^2 + \frac{1}{\lambda+1}(1+x)\right)\right)} e^{-(\theta-\lambda)x}$$

Therefore,

$$\log[\Lambda] = \log\left(\frac{\theta^{c_1+2}\left(\frac{\beta(c_2+2)!}{2} + \frac{(\theta c_2! + (c_2+1)!)}{\lambda+1}\right)}{\lambda^{c_2+2}\left(\frac{\alpha(c_1+2)!}{2} + \frac{(\theta c_1! + (c_1+1)!)}{\theta+1}\right)}\right) + \log\left(x^{c_1}\left(\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right)\right) - \log\left(x^{c_2}\left(\frac{1}{2}\beta\lambda x^2 + \frac{1}{\lambda+1}(1+x)\right)\right) - (\theta-\lambda)x$$

Differentiating with respect to x, we get.

$$\frac{\partial \log[\Lambda]}{\partial x} = \left(\frac{c_1 x^{c_1+1}}{x^{c_1}}\left(\frac{4}{\alpha\theta x} + \frac{\theta+1}{1+x}\right)\right) - \left(\frac{c_2 x^{c_2+1}}{x^{c_2}}\left(\frac{4}{\beta\lambda x} + \frac{\lambda+1}{1+x}\right)\right) + (\lambda - \theta)$$

Hence, $\frac{\partial \log[\Lambda]}{\partial x} < 0$ if $\alpha < \beta, \theta < \lambda\ and, c_1 < c_2$

XI. ENTROPIES

In this section, we derived the Shannon entropy, Rényi entropy, and Tsallis entropy from the weighted Loai distribution.

It is well known that entropy and information can be considered measures of uncertainty or the randomness of a probability distribution. It is applied in many fields, such as engineering, finance, information theory, and biomedicine. The entropy functionals for probability distribution were derived on the basis of a variational definition of uncertainty measure.

9.1 Shannon Entropy

Shannon entropy of the random variable X such that weighted Loai distribution is defined as

$$S_\lambda = -\int_0^\infty f(x)\log\big(f(x)\big)dx \qquad ;\lambda > 0, \lambda \neq 1$$

$$S_\lambda = -\int_0^\infty f_w(x;\alpha,\theta,c)\log\big(f_w(x;\alpha,\theta,c)\big)dx$$

$$S_\lambda = -\int_0^\infty \frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c! + (c+1)!)}{\theta+1}\right)} x^c \left(\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right)e^{-\theta x} \times \log\left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c! + (c+1)!)}{\theta+1}\right)} x^c \left(\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right)e^{-\theta x}\right)dx$$

9.2 Rényi Entropy

Entropy is defined as a random variable X is a measure of the variation of the uncertainty. It is used in many fields, such as engineering, statistical mechanics, finance, information theory, biomedicine, and economics. The entropy measure is the Rényi of order which is defined as

$$R_\lambda = \frac{1}{1-\lambda}\log\int_0^\infty [f(x)]^\lambda \, dx \qquad ;\lambda > 0, \lambda \neq 1$$

$$R_\lambda = \frac{1}{1-\lambda}\log\int_0^\infty [f_w(x;\alpha,\theta,c)]^\lambda \, dx$$

$$R_\lambda = \frac{1}{1-\lambda}\log\int_0^\infty \left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c! + (c+1)!)}{\theta+1}\right)} x^c \times \left(\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right)e^{-\theta x}\right)^\lambda dx$$

$$R_\lambda = \frac{1}{1-\lambda}\log\left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c! + (c+1)!)}{\theta+1}\right)}\right)^\lambda \times \int_0^\infty x^{c\lambda}\left(\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right)^\lambda e^{-\theta\lambda x}dx$$

Using Binomial expansion, we get

$$\left(\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right)^\lambda = \left(\frac{\alpha\theta}{2} x^2 + \frac{1}{\theta+1}(1+x)\right)^\lambda$$

$$\left(\frac{\alpha\theta}{2} x^2 + \frac{1+x}{\theta+1}\right)^\lambda = \sum_{i=0}^{\lambda}\binom{\lambda}{i}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i} x^{2(\lambda-k)}\left(\frac{1}{\theta+1}\right)^i (1+x)^i$$

$$\left(\frac{\alpha\theta}{2} x^2 + \frac{1+x}{\theta+1}\right)^\lambda = \sum_{i=0}^{\lambda}\binom{\lambda}{i}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i}\left(\frac{1}{\theta+1}\right)^i x^{2(\lambda-i)} (1+x)^i$$

$$(1+x)^i = \sum_{j=0}^{i}\binom{i}{j} 1^{i-j} x^j$$

$$(1+x)^i = \sum_{i=0}^{\lambda}\binom{\lambda}{i}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i}\left(\frac{1}{\theta+1}\right)^i x^{2(\lambda-i)} x^j$$

$$(1+x)^i = \sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i}\left(\frac{1}{\theta+1}\right)^i x^{2(\lambda-i)} x^j$$

$$R_\lambda = \frac{1}{1-\lambda}\log\left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c! + (c+1)!)}{\theta+1}\right)}\right)^\lambda \int_0^\infty \sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i}\left(\frac{1}{\theta+1}\right)^i x^{c\lambda}x^{2(\lambda-i)}x^j e^{-\lambda\theta x}\, dx$$

$$R_\lambda = \frac{1}{1-\lambda}\log\left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\right)^{\lambda}\sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i}\left(\frac{1}{\theta+1}\right)^{i} \times \int_0^\infty x^{c\lambda+2(\lambda-i)+j}\, e^{-\lambda\theta x}\, dx$$

$$R_\lambda = \frac{1}{1-\lambda}\log\left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\right)^{\lambda}\sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i}\left(\frac{1}{\theta+1}\right)^{i}\left(\frac{1}{\lambda\theta}\right)^{c\lambda+2(\lambda-i)+j+1} \times \Gamma(c\lambda+2(\lambda-i)+j+1)$$

9.3 Tsallis Entropy

The Boltzmann-Gibbs (B-G) statistical properties initiated by Tsallis have received a great deal of attention. This generalization of (B-G) statistics was first proposed by introducing the mathematical expression of Tsallis entropy (Tsallis, (1988) for continuous random variables, which is defined as

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\int_0^\infty [f(x)]^\lambda\, dx\right] \qquad ;\lambda>0,\lambda\neq 1$$

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\int_0^\infty [f_w(x;\alpha,\theta,c)]^\lambda\, dx\right]$$

$$T_\lambda = \frac{1}{\lambda-1}\left(1-\int_0^\infty\left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}x^c \times\left(\frac{1}{2}\alpha\theta\, x^2+\frac{1}{\theta+1}(1+x)\right)e^{-\theta x}\right)^{\lambda} dx\right)$$

$$T_\lambda = \frac{1}{\lambda-1}\left(1-\left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\right)^{\lambda} \times\int_0^\infty x^{c\lambda}\left(\frac{1}{2}\alpha\theta\, x^2+\frac{1}{\theta+1}(1+x)\right)^{\lambda} e^{-\theta\lambda x}\, dx\right) \qquad (12)$$

Using Binomial expansion, we get

$$\left(\frac{1}{2}\alpha\theta\, x^2+\frac{1}{\theta+1}(1+x)\right)^{\lambda} = \left(\frac{\alpha\theta}{2}\, x^2+\frac{1}{\theta+1}(1+x)\right)^{\lambda}(1+x)^i$$

$$= \sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i}\left(\frac{1}{\theta+1}\right)^{i} x^{2(\lambda-i)}\, x^j \qquad (13)$$

Substituting equation (13) in (12) we get,

$$T_\lambda = \frac{1}{\lambda-1}\left(1-\left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\right)^{\lambda}\sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i}\left(\frac{1}{\theta+1}\right)^{i} \times\int_0^\infty x^{c\lambda+2(\lambda-i)+j}\, e^{-\lambda\theta x}\, dx\right)$$

$$T_\lambda = \frac{1}{\lambda-1}\left(1-\left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2}+\frac{(\theta c!+(c+1)!)}{\theta+1}\right)}\right)^{\lambda}\sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\left(\frac{\alpha\theta}{2}\right)^{\lambda-i}\left(\frac{1}{\theta+1}\right)^{i}\left(\frac{1}{\lambda\theta}\right)^{c\lambda+2(\lambda-i)+j+1} \times\Gamma(c\lambda+2(\lambda-i)+j+1)\right)$$

XII. ESTIMATIONS OF PARAMETER

In this section, the maximum likelihood estimates and Fisher's information matrix of the weighted Loai distribution parameter is given.

12.1 Maximum Likelihood estimation (MLE)and Fisher's Information Matrix

Consider $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from the weighted Loai distribution with parameter α, θ and c the likelihood function, which is defined as

$$L(x; \alpha, \theta, c) = \prod_{i=1}^{n} f_w(x_i; \alpha, \theta, c)$$

$$= \prod_{i=1}^{n} \left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} x_i^c \left(\frac{1}{2}\alpha\theta x_i^2 + \frac{1}{\theta+1}(1+x_i)\right) e^{-\theta x_i} \right)$$

$$= \left(\frac{\theta^{c+2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \right)^n \prod_{i=1}^{n} \left(x_i^c \left(\frac{1}{2}\alpha\theta x_i^2 + \frac{1}{\theta+1}(1+x_i)\right) e^{-\theta x_i} \right)$$

Then, the log-likelihood function is

$$\ell = \log L = n(c+2)\log(\theta) - n\log\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right) + c\sum_{i=0}^{n} \log x_1 + \sum_{i=0}^{n} \left(\frac{1}{2}\alpha\theta x_i^2 + \frac{1}{\theta+1}(1+x_i)\right) - \theta\sum_{i=0}^{n} x_i \quad (14)$$

Deriving (14) partially with respect to α, θ and c we have.

$$\frac{\partial \log L}{\partial \theta} = \frac{n(c+2)}{\theta} - n\left(\frac{\frac{c! - (c+1)!}{(\theta+1)^2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \right) + \sum_{i=1}^{n} \left(\frac{\frac{\alpha x_i^2}{2} - \frac{1+x_i}{\theta+1}}{\left(\frac{1}{2}\alpha\theta x_i^2 + \frac{1}{\theta+1}(1+x_i)\right)} \right) - \sum_{i=1}^{n} x_i = 0 \quad (15)$$

$$\frac{\partial \log L}{\partial \alpha} = -n\left(\frac{\frac{(c+2)!}{2}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \right) + \sum_{i=1}^{n} \left(\frac{\frac{\theta x_i^2}{2}}{\left(\frac{1}{2}\alpha\theta x_i^2 + \frac{1}{\theta+1}(1+x_i)\right)} \right) = 0 \quad (16)$$

$$\frac{\partial \log L}{\partial c} = n\log(\theta) - n\left(\frac{\frac{\alpha\Psi(c+2)}{2} + \frac{\theta\Psi(c) + \Psi(c+1)}{\theta+1}}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)} \right) + \sum_{i=1}^{n} \log(x_i) = 0 \quad (17)$$

Where $\Psi(.)$ Is the digamma function.

The equation (15), (16) and (17) gives the maximum likelihood estimation of the parameters for the weighted Loai distribution. However, the equation cannot be solved analytically, thus we solved numerically using R programming with data set.

To obtain confidence interval we use the asymptotic normality results. We have that if $\hat{\lambda} = (\hat{\theta}, \hat{\alpha}, \hat{c})$ denotes the MLE of $\lambda = (\theta, \alpha, c)$ we can state the results as follows:

$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N_2(0, I^{-1}(\lambda))$

Where $I(\lambda)$ is Fisher's Information Matrix. i.e.,

$$I(\lambda) = -\frac{1}{n} \begin{pmatrix} E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right] & E\left[\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right] & E\left[\frac{\partial^2 \log L}{\partial \theta \partial c}\right] \\ E\left[\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right] & E\left[\frac{\partial^2 \log L}{\partial \alpha^2}\right] & E\left[\frac{\partial^2 \log L}{\partial \alpha \partial c}\right] \\ E\left[\frac{\partial^2 \log L}{\partial c \partial \alpha}\right] & E\left[\frac{\partial^2 \log L}{\partial c \partial \theta}\right] & E\left[\frac{\partial^2 \log L}{\partial c^2}\right] \end{pmatrix}$$

$$E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right] = -\frac{n(c+2)}{\theta^2} - \frac{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)\left(\frac{2(c! - (c+1)!)}{(\theta+1)^3}\right) - \left(\frac{c! - (c+1)!}{(\theta+1)^2}\right)^2}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)^2}$$

$$- \sum_{i=1}^{n} \frac{\left(\frac{1}{2}\,\alpha\theta\, x_i^2 + \frac{1}{\theta+1}(1+x_i)\right)\frac{2(1+x)}{(\theta+1)^2} - \left(\frac{\alpha x_i^2}{2} - \frac{1+x}{(\theta+1)^2}\right)^2}{\left(\frac{1}{2}\,\alpha\theta\, x_i^2 + \frac{1}{\theta+1}(1+x_i)\right)^2}$$

$$E\left[\frac{\partial^2 \log L}{\partial \alpha^2}\right] = -n\left(\frac{\left(\frac{(c+2)!}{2}\right)^2}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)^2}\right) - \sum_{i=1}^{n}\left(\frac{\left(\frac{\theta x_i^2}{2}\right)^2}{\left(\frac{1}{2}\,\alpha\theta\, x_i^2 + \frac{1}{\theta+1}(1+x_i)\right)^2}\right)$$

$$E\left[\frac{\partial^2 \log L}{\partial c^2}\right] = n\left(\frac{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)\left(\frac{\alpha\Psi'(c+2)}{2} + \frac{\theta\Psi'(c) + \Psi'(c+1)}{\theta+1}\right) - \left(\frac{\alpha\Psi(c+2)}{2} + \frac{\theta\Psi(c) + \Psi(c+1)}{\theta+1}\right)^2}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)^2}\right)$$

$$E\left[\frac{\partial^2 \log L}{\partial \alpha \partial c}\right] = \frac{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)\left(\frac{\Psi(c+2)}{2}\right) - \left(\frac{\alpha\Psi(c+2)}{2} + \frac{\theta\Psi(c) + \Psi(c+1)}{\theta+1}\right)\left(\frac{\alpha(c+2)!}{2}\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)^2}$$

$$E\left[\frac{\partial^2 \log L}{\partial \theta \partial c}\right] = \frac{n}{\theta} - \frac{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)\left(\frac{\Psi(c) + \Psi(c+1)}{(\theta+1)^2}\right) - \left(\frac{\alpha\Psi(c+2)}{2} + \frac{\theta\Psi(c) + \Psi(c+1)}{\theta+1}\right)\left(\frac{c! - (c+1)!}{(\theta+1)^2}\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)^2}$$

$$\left[\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right] = -n\left(\frac{\left(\frac{c! - (c+1)!}{(\theta+1)^2}\right)\left(\frac{(c+2)!}{2}\right)}{\left(\frac{\alpha(c+2)!}{2} + \frac{(\theta c! + (c+1)!)}{\theta+1}\right)^2}\right)$$

$$- \sum_{i=1}^{n}\left(\frac{\left(\frac{1}{2}\,\alpha\theta\, x_i^2 + \frac{1}{\theta+1}(1+x_i)\right)\left(\frac{x_i^2}{2}\right) - \left(\frac{\alpha x_i^2}{2} - \frac{1+x_i}{(\theta+1)^2}\right)\left(\left(\frac{\theta x_i^2}{2}\right)\right)}{\left(\frac{1}{2}\,\alpha\theta\, x_i^2 + \frac{1}{\theta+1}(1+x_i)\right)^2}\right)$$

Where $\Psi(.)'$ is the first order derivative of digamma function. Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence interval for θ, α, c.

XIII. APPLICATIONS

Dat set 1: This data consists of the life time (in years) of 40-blood cancer (leukemia) patients from one of ministry of health hospitals in Sdudhi Arabia reported in [01]. This actual data is

0.315,0.496,0.616,1.145,1.208,1.263,1.414,2.025,2.036,2.162,2.211,2.370,2.532,2.693,2.805,2.910,2.912,3.192, 3.263,3.348,3.427,3.499,3.534,3.767,3.751,3.858,3.986,4.049,4.244,4.323, 4.381, 4.392,4.397,4.647,4.753,4.929,4.973,5.074,5.381.

Data set 2: The data under consideration are the life times of 20 leukemia patients who were treated by a certain drug [15]. The data are

1.013,1.034,1.109,1.226,1.509,1.533,1.563,1.716,1.929,1.965,2.061,2.344,2.546,2.626,2.778,2.951,3.413,4.118, 5.136

Data set 3: [20] Consider a simulated data represents the survival times (in days) of 73 patients who diagnosed with acute bone cancer, as follows

0.09,0.76,1.81,1.10,3.72,0.72,2.49,1.00,0.53,0.66,31.61,0.60,0.20,1.61,1.88,0.70,1.36,0.43,3.16,1.57,4.93,11.07,1.63,1.39,4.54,3.12,76.01,1.92,0.92,4.04,1.16,2.26,0.20,0.94,1.82,3.99,1.46,2.75,1.38,2.76,1.86,2.68,1.76,0.67,1.29,1.56,2.83,0.71,1.48,2.41,0.66,0.65,2.36,1.29,13.75,0.67,3.70,0.76,3.63,0.68,2.65,0.95,2.30,2.57,0.61,3.93,1.56,1.29,9.94,1.67,1.42,4.18,1.37

Data set 4: The data set is reported by [05] and, which corresponds to the survival times (in years) of a group of patients group of patients given by chemotherapy treatment alone.

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.581, 1.589, 2.178, 2.343, 2.461, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

To compare to the goodness of fit of the fitted distribution, the following criteria: Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Akaike Information Criteria Corrected (AICC) and $-2\log L$.
AIC, BIC, AICC and $-2\log L$ can be evaluated by using the formula as follows.

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L \ and\ AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$

Where, $k =$ number of parameters, n sample size and $-2\log L$ is the maximized value of loglikelihood function.

Table 1; MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 1

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Weighted Loai distribution	$\hat{\boldsymbol{\alpha}} = \mathbf{0.0010000}$ (**NaN**) $\hat{\boldsymbol{\theta}} = \mathbf{1.8461208}(\mathbf{0.2105503})$ $\hat{\boldsymbol{c}} = \mathbf{3.9551765}(\mathbf{0.5186100})$	**84.28011**	**90.28011**	**95.2708**	**90.9477**
Loai distribution	$\hat{\alpha} = 21.37415661$ (51.96074056 $\hat{\theta} = 0.9356678$ (0.09885747)	144.0757	148.0757	151.4025	148.7423
Aradhana	$\hat{\theta} = 0.75060122$ (0.07108124)	149.4283	151.4283	153.0918	151.5335
Ishita	$\hat{\theta} = 0.80668240$ (0.06521656)	147.9967	149.9967	151.6603	150.1019
Akshaya	$\hat{\theta} = 0.98152890$ (0.07981806)	144.7945	150.7945	152.4582	152.8997
Shanker	$\hat{\theta} = 0.54972161$ (0.05806214)	144.7945	155.9545	157.6181	156.0597
Rama	$\hat{\theta} = 1.10146523$ (0.08055189)	143.3158	149.3158	151.1023	151.4210
Exponential	$\hat{\theta} = 0.31893857$ (0.05107054)	167.1353	169.1353	170.7988	169.0405
Lindley	$\hat{\theta} = 0.2577071$ (0.06161721)	156.5028	158.5028	160.1664	158.6080
Akash	$\hat{\theta} = 0.80168363$ (0.07120997)	149.0561	151.0561	152.7196	151.1613

Table 2; MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 2

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Weighted Loai distribution	$\hat{\boldsymbol{\alpha}} = \mathbf{0.0010000}$ ($\boldsymbol{NaN}$) $\hat{\boldsymbol{\theta}} = \mathbf{3.0762093}$ ($\mathbf{0.59799806}$) $\hat{\boldsymbol{c}} = \mathbf{5.3324990}$ ($\mathbf{1.1443841}$)	**18.84248**	**24.84248**	**27.6759**	**26.44248**
Loai distribution	$\hat{\alpha} = 2770.243673$ (19373.500352) $\hat{\theta} = 1.338780$ (0.177363)	53.59238	57.59238	59.48126	59.19238
Aradhana	$\hat{\theta} = 0.985545$ (0.135948)	60.60053	62.60053	63.54497	62.8227
Ishita	$\hat{\theta} = 0.9975990$ (0.1134076)	62.74297	64.74297	65.68741	64.9651

Akshaya	$\hat{\theta} = 1.2738546$ (0.1506672)	58.05546	60.05546	60.9999	60.2776
Shanker	$\hat{\theta} = 0.7124395$ (0.10777871)	63.08856	65.08856	66.033	65.3107
Rama	$\hat{\theta} = 1.3784229$ (0.1415338)	62.41991	64.41991	65.36435	64.6421
Exponential	$\hat{\theta} = 0.4463246$ (0.1023934)	68.65501	70.65501	71.59945	70.8772
Lindley	$\hat{\theta} = 0.7076860$ (0.1200725)	64.02158	66.02158	66.96602	66.2438
Akash	$\hat{\theta} = 0.0297001$ (0.1317933)	62.69158	64.69158	65.63602	64.9138

Table 3; MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 3

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Weighted Loai distribution	$\hat{\alpha} = 0.0010000$ $\hat{\theta} = 0.4892185$ (**NaN**) $\hat{c} = 0.0010006$ (**NaN**)	**329.8993**	**336.8983**	**342.7469**	**337.2461**
Loai distribution	$\hat{\alpha} = 0.00300600(0.041126948)$ $\hat{\theta} = 0.46603344(0.041637511)$	365.8571	369.8571	374.438	370.2049
Aradhana	$\hat{\theta} = 0.66657919$ (0.04589789)	405.5844	407.5844	409.8748	407.6407
Ishita	$\hat{\theta} = 0.7626730$ (0.0449009)	425.5164	427.5164	429.8069	427.5727
Akshaya	$\hat{\theta} = 0.87506525$ (0.05179087)	450.401	452.401	454.6915	452.4573
Shanker	$\hat{\theta} = 0.51244299$ (0.03919024)	373.2109	375.2109	377.5014	375.2672
Rama	$\hat{\theta} = 0.99007435$ (0.05367476)	483.9594	485.9594	488.2499	486.0157
Exponential	$\hat{\theta} = 0.27638027$ (0.03234744)	333.7534	375.7534	338.0438	375.8097
Lindley	$\hat{\theta} = 0.46503064$ (0.03949368)	365.8631	368.8631	370.1536	368.9294
Akash	$\hat{\theta} = 0.71628684$ (0.04656348)	419.7666	421.7666	424.051	421.8230

Table 4; MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 4

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Weighted Loai distribution	$\hat{\alpha} = 0.0010000$ $\hat{\theta} = 1.1095679\ (0.5080874)$ $\hat{\alpha} = 0.0010000\ (0.4764376)$	**95.14074**	**101.1408**	**106.7544**	**101.6862**
Loai distribution	$\hat{\alpha} = 0.1170399$ (0.3693474) $\hat{\theta} = 1.2261028$ (0.3305175)	122.565	126.565	130.3075	127.1104
Aradhana	$\hat{\theta} = 1.4963630$ (0.1351993)	124.5748	126.5748	128.446	126.6657
Ishita	$\hat{\theta} = 1.4066951$ (0.1027368)	122.1315	128.1315	126.0027	128.2224
Akshaya	$\hat{\theta} = 1.8857589$ (0.1463088)	127.6693	129.6693	131.5405	129.7602
Shanker	$\hat{\theta} = 1.1087323$ (0.1066882)	122.5482	126.5482	127.4194	126.6392
Rama	$\hat{\theta} = 1.8610360$ (0.1239909	123.0723	128.0723	130.9435	128.1632

Exponential	$\hat{\theta} = 0.7607586$ (0.1098058)	122.2503	127.2505	128.1215	127.3415
Lindley	$\hat{\theta} = 1.119663$ (0.123191)	122.6538	130.6538	129.525	130.7457
Akash	$\hat{\theta} = 1.484596$ (0.124779)	122.7795	134.7795	131.6507	134.8705

From table 1, 2, 3, and 4 it can be clearly observed and seen from the results that the weighted Loai distribution have the lesser AIC, BIC, AICC, -2log L, and values as compared to the Loai, Aradhana, Ishita, Akshaya, Shanker, Rama, Exponential, Lindley, Akash distributions, which indicates that the weighted Loai distribution better fits than the Loai, Aradhana, Ishita, Akshaya, Shanker, Rama, Exponential, Lindley, Akash distributions. Hence, it can be concluded that the weighted Loai distribution leads to a better fit over the other distributions.

XIV. CONCLUSIONS

Selecting a suitable model for fitting survival data has been a major concern among researchers. One of the most popular distributions for real-time data is the Loai distribution. In this paper, the Loai distribution is extended to provide a new distribution called the weighted Loai distribution to model life-time data. They proposed cdf and pdf of the weighted Loai distribution in Section 2. Considered the mathematical and statistical properties of the derived distribution. From the graphs drawn from pdf and cdf-derived distributions. The hazard rate function, survival function, and their graphs for the new distribution are given in Section 3. The expressions for finding the mean and median are given in sections 5 and 6. The expression for its r^{th} moments of derived distribution is given in 4. We have also derived the entropy and the pdf of its r^{th} order statistics in 11 and 7. The use of statistical distributions in medical research is critical and can have a significant impact on the general public's health, particularly for cancer patients. thus, the applications of this distribution to certain real data sets that describe the survival of some cancer patients provide a demonstration of the utility of this distribution. The method of maximum likelihood estimation to estimate its parameters is discussed in Section 12. Moreover, the derived distribution is applied to four real data sets and compared with the other well-known distribution in Section 13. Results show that the weighted Loai distribution provides a better fit than other well-known distributions.

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