International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 11 Issue 4 || July - August, 2023 || PP 01-16

Analysis of Disease Propagation Models using Fixed Point Theorem and G-metric Spaces

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Abstract:
The study investigates the existence and uniqueness of solutions for Fisher's equations in disease propagation models by utilizing the principles of fixed point theorem, Morse theorem, and G-metric spaces. Fisher's equations, which describe the spatial-temporal dynamics of populations, are widely employed in modelling disease spread. By formulating the equations as a fixed point problem, the application of the contraction mapping principle establishes the existence and uniqueness of a weak solution. By characterizing critical points and analysing their stability through the Morse index, insights into the spreading dynamics of diseases can be gained. By considering Fisher's equations within the context of G-metric spaces, the study explores the properties of generalized metrics and their implications for disease propagation models. By imposing suitable conditions on the coefficients of Fisher's equations, different scenarios of disease propagation and their effects on the solutions are examined.

Keywords: *Fisher's equations, Disease propagation, Fixed point theorem, Morse theorem, G-metric spaces, Contraction mapping, Critical points, Stability analysis, Spatial-temporal dynamics.*

Date of Submission: 12-07-2023 Date of acceptance: 29-07-2023

I. Introduction:

The Fisher equation, also known as the Fisher-Kolmogorov-Petrovsky-Piskunov equation, is a well-known partial differential equation that has applications in various fields, including population dynamics, disease propagation, and ecological modelling. For ecological and diffusion models the Fisher-KPP equations is described as in [6] ,

$$u_t = Du_{xx} + ru\left(1 - \frac{u}{k}\right) \qquad [1.1]$$

In this study, we focus on investigating the existence and uniqueness of solutions for the Fisher equation using the G-metric method and the fixed point theorem, specifically the contraction mapping principle [14]. The G-metric method provides a framework for studying metric spaces that have properties resembling those of conventional metrics. This approach allows us to extend the notion of distance and convergence in traditional metric spaces to a more flexible and adaptable setting, enabling the investigation of unique solutions for the Fisher equation under varying conditions. The fixed point theorem provides a powerful tool to establish the existence and stability of fixed points in a mapping, which is crucial for understanding the behaviour of solutions in the context of the Fisher equation [3] ,[5] .

In this study we also analysed the effects of different conditions of the carrying capacity (K) and diffusion rate (D) [6] . The carrying capacity represents the maximum population size that the environment can support. Higher values of K indicate a larger potential population size, while lower values imply more restricted growth [10],[2]. The diffusion rate, on the other hand, determines the rate at which individuals disperse and spread across different locations. The diffusion rate, on the other hand, determines the rate at which individuals disperse and spread across different locations.

To analyse the dynamics of the system and understand the behaviour of the solutions, we employ the Morse theorem and it provides a powerful tool for studying the critical points of a dynamical system and characterizing their stability properties [1]. By examining the critical points of the Fisher equation, we can gain insights into the long-term behaviour of the population and identify stable or unstable equilibrium points.

II. Fixed Point theorem:

The fixed point theorem states that under certain conditions, there exists a unique point in the model where the population densities or other ecological variables remain constant over time. The contraction mapping principle, on the other hand, provides a mathematical framework to prove the existence of a unique fixed point in a given space [9]. It requires the mapping function to satisfy a contraction property, which ensures that nearby points in the space are mapped closer together. This contraction property guarantees convergence to a fixed point and allows us to establish the existence and uniqueness of equilibrium solutions in ecological models as given in [8] ,[9] ,[5].

2.1 Preliminaries:

Theorem 1 (Continuity and Sequence) [11]:
Suppose $f: X \to Y$ is a continuous function between metric spaces and let $\{x_n\}$ be a sequence of points of X which converges to $x \in X$. Then the sequence $\{f(x_n)\}$ must converge to f(x).
Theorem 2 (Continuity and open sets):
A function $f: X \to Y$ between metric spaces is continuous if and only if $f^{-1}(U)$ is open in X for each set U which is open in Y.
Theorem 3 (Lipschitz Continuity):
i) Every Lipschitz continuous function is continuous.
Proof. Suppose $f : X \to Y$ is a Lipschitz continuous function. To show that f is continuous at all points $x \in X$, let $\varepsilon > 0$ be given. Since f is Lipschitz continuous, we have

$$d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)$$

for some $L \geq 0$. When $L > 0$, we can take $\delta = \varepsilon/L$ to find that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)$$
$$\Rightarrow d_Y(f(x), f(y)) < L \cdot \delta = \varepsilon.$$

When $L = 0$, one always has $d_Y(f(x), f(y)) \leq 0 < \varepsilon$, so the choice of δ is irrelevant. Thus, f is continuous at x in any case.
ii) If a function $f : [a, b] \to R$ is differentiable and its derivative is bounded, then f is Lipschitz continuous on [a, b].
Proof. Suppose $|f'(x)| \leq M$ for all $x \in [a, b]$ and let $x, y \in [a, b]$ be arbitrary. Using the mean value theorem, we can then write

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y|$$

for some c between x and y. This obviously gives

$$|f(x) - f(y)| \leq M \cdot |x - y|$$

for all $x, y \in [a, b]$ and so f is Lipschitz continuous on [a, b].
Theorem 4 (Pointwise and uniform convergence):
i) To say that $f_n(x) \to f(x)$ pointwise is to say that, $|f_n(x) - f(x)| \to 0$ as $n \to \infty$.
Proof. By definition, to say that $f_n(x) \to f(x)$ pointwise is to say that, given any $\varepsilon > 0$ there exists an integer N such that
$|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$.
This is the case if and only if $|f_n(x) - f(x)| \to 0$ as $n \to \infty$.
ii) To say that $f_n \to f$ uniformly on X is to say that $\sup_{x\in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.
Proof. Let $M_n = \sup_{x\in X}|f_n(x) - f(x)|$ for convenience. Suppose that $M_n \to 0$ as $n \to \infty$ and let $\varepsilon > 0$ be given. Then there is an integer N such that $M_n < \varepsilon$ for all $n \geq N$, so we also have

$$|f_n(x) - f(x)| \leq M_n < \varepsilon$$

for all $n \geq N$ and all $x \in X$. In particular, $f_n \to f$ uniformly on X. Conversely, suppose $f_n \to f$ uniformly on X and let $\varepsilon > 0$ be given. Then there is an integer N such that $|f_n(x) - f(x)| < \varepsilon/2$ for all $n \geq N$ and all $x \in X$. Taking the supremum over all $x \in X$, we get $M_n \leq \varepsilon/2 < \varepsilon$ for all $n \geq N$, so $M_n \to 0$ as $n \to \infty$.
Theorem 5 (Uniform limit of continuous functions):
The uniform limit of continuous functions is continuous: if each f_n is continuous and $f_n \to f$ uniformly on X, then f is continuous on X.
Theorem 6 (Convergent implies Cauchy):
In a metric space, every convergent sequence is a Cauchy sequence.
Proof. Suppose that $\{x_n\}$ is a sequence which converges to x and let $\varepsilon > 0$ be given. Then there exists an integer N such that
$d(x_n, x) < \varepsilon/2$ for all $n \geq N$.
Using this fact and the triangle inequality, we conclude that
$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \varepsilon$
for all $m, n \geq N$. This shows that the sequence is Cauchy.
Theorem 7 (Cauchy implies bounded):
In a metric space, every Cauchy sequence is bounded.
Theorem 8 (Cauchy sequence with convergent subsequence):
Suppose (X,d) is a metric space and let $\{x_n\}$ be a Cauchy sequence in X that has a convergent subsequence. Then $\{x_n\}$ converges itself.

Proof. Suppose that $\{x_n\}$ is Cauchy and $\{x_{n_k}\}$ converges to x. We claim that $\{x_n\}$ converges to x as well. Let $\varepsilon > 0$ be given. Then there exist integers N_1, N_2 such that
$d(x_m, x_n) < \varepsilon/2$ for all $m, n \geq N_1$,
$d(x_{n_k}, x) < \varepsilon/2$ for all $n_k \geq N_2$.
Set $N = \max\{N_1, N_2\}$ and fix some $n_k \geq N$. Then we have
$d(x_m, x) \leq d(x_m, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$
for all $m \geq N$ and this implies that $x_m \to x$ as $m \to \infty$.
Theorem 9 (Fixed point theorem):
If $f : X \to X$ is a contraction on a complete metric space X, then f has a unique fixed point, namely a unique point x with $f(x) = x$.
Proof: First, we prove uniqueness. Suppose that $x \neq y$ are both fixed points. Since f is a contraction, we then have
$d(x, y) = d(f(x), f(y)) \leq \alpha \cdot d(x, y)$
for some $0 \leq \alpha < 1$. This leads to the contradiction $\alpha \geq 1$.
It remains to show existence. Let $x \in X$ be arbitrary and define a sequence by setting $x_1 = x$ and $x_{n+1} = f(x_n)$ for each $n \geq 1$. If this sequence is actually Cauchy, then it converges by completeness and its limit y is a fixed point of f because
$y = \lim_{n\to\infty} x_n \Rightarrow f(y) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = y$.
It remains to show that our sequence is Cauchy. Since f is a contraction and $x_{n+1} = f(x_n)$ for each n, we have
$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + \ldots + d(x_{n+k-1}, x_{n+k})$

$$= \sum_{i=n}^{n+k-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{n+k-1} \alpha^{i-1}.d(x_1, x_2) \leq \sum_{i=n}^{\infty} \alpha^{i-1}.d(x_1, x_2) = \frac{\alpha^{n-1}}{1-\alpha} \cdot d(x1, x2).$$

The right hand side goes to zero as $n \to \infty$, so the same is true for the left hand side. This shows that our sequence is Cauchy.

Theorem 10 (Existence and uniqueness of solutions):
Consider an initial value problem of the form

$$y'(t) = f(t, y(t)), \quad y(0) = y_0.$$

If f is continuous in t and Lipschitz continuous in y, then there exists a unique solution y(t) which is defined on $[0, \varepsilon]$ for some $\varepsilon > 0$.
Proof. It is easy to see that y(t) is a solution if and only if

$$y(t) = y_0 + \int_0^t f(s, y(s))\, ds.$$

Let us denote the right hand side by A(y(t)). Then every solution corresponds to a fixed point of A, so it suffices to show that A is a contraction on a complete metric space X. In fact, $X = C[0, \varepsilon]$ is complete with respect to the d_∞ metric for any $\varepsilon > 0$.
To show that A is a contraction, we note that

$$|f(s, y(s)) - f(s, z(s))| \leq L|y(s) - z(s)| \leq L\, d_\infty(y, z).$$

Fix some $0 < \varepsilon < 1/L$ and let $y(t), z(t) \in C[0, \varepsilon]$. We then have

$$|A(y(t)) - A(z(t))| \leq L\, d_\infty(y, z)\int_0^t ds \leq \varepsilon L\, d_\infty(y, z)$$

for all $t \in [0, \varepsilon]$ and this implies that A is a contraction, indeed.

2.2 Contraction Mapping Principle
We initiate our exploration of nonlinear partial differential equations (PDEs) taking reference from [9] with the following fundamental abstract result.
Theorem 1 (Contraction Mapping Principle):
Let (X,d) be a complete metric space, and consider a mapping $A: X \to X$. Assume that A is a contraction on X, meaning there exists a constant $y \in (0,1)$ such that

$$d(A(x),A(y)) \leq \gamma d(x,y) \text{ for all } x,y \in X. \qquad [2.1.1]$$

Under these conditions, there exists a unique $x0 \in X$ such that $A(x0) = x0$, signifying that A possesses a unique fixed point within X.

Theorem 2:
Let (X,d) be a complete metric space, and let $A : X \to X$ be a mapping. Additionally, assume that there exists an element $a \in X$ and a positive radius $r > 0$ such that:
(i) The ball $B(a,r) := \{x \in X : d(x,a) < r\}$ is an invariant set for A, meaning $A : B(a,r) \to B(a,r)$.
(ii) The map A is a contraction on the ball B(a,r), i.e., there exists a constant $y \in (0,1)$ such that

$$d(A(x),A(y)) \leq \gamma d(x,y) \text{ for all } x,y \in B(a,r). \qquad [2.1.2]$$

Under these conditions, there exists a fixed point $x0 \in B(a,r)$ such that $A(x0) = x0$, and furthermore, this fixed point is unique within the ball B(a,r).

The proof of the aforementioned theorem follows a similar approach to that of the Contraction Mapping Principle and is therefore omitted here.

2.3 Proof:

Let $\Omega \subset R^3$ be open and bounded with smooth boundary in Fisher's equation,

$u_t = u_{xx} + u(1 - u)$, $g(t) < x < h(t)$, $t > 0$, in Ω

$u(t, g(t)) = 0$, $g(t) = -\mu u_x(t, g(t))$, $t > 0$, on $\partial\Omega$

$u(t, h(t)) = 0$, $h(t) = -\mu u_x(t, h(t))$, $t > 0$, on $\partial\Omega$

$u(\theta, x) = \varphi(\theta, x)$, $\theta \in [-\tau, 0]$, $x \in X$ on $\partial\Omega$

and let X be the function space defined as:

$$X = C([-\tau, 0], L^2(g(\theta), h(\theta))) \cap C^1([-\tau, 0], H^1(g(\theta), h(\theta))) \qquad [2.3.1]$$

Here, $C([-\tau, 0], L^2(g(\theta), h(\theta)))$ denotes the space of continuous functions on the interval $[-\tau, 0]$ with values in $L^2(g(\theta), h(\theta))$, and $C^1([-\tau, 0], H^1(g(\theta), h(\theta)))$ represents the space of continuously differentiable functions on $[-\tau, 0]$ with values in $H^1(g(\theta), h(\theta))$ and $g(\theta), h(\theta) \in \Omega$.

Equipped with suitable norms, this function space provides the necessary properties for the existence and uniqueness of the weak solution. Specifically, we consider the H^1 norm for the space $C^1([-\tau, 0], H^1(g(\theta), h(\theta)))$ and the L^2 norm for the space $C([-\tau, 0], L^2(g(\theta), h(\theta)))$.

By Sobolev embedding, we have the continuity estimate as given in [9]:

$$\|v\|_{L^\infty([-\tau,0],L^2(g(\theta),h(\theta)))} \leq C\, \|v\|_{H^1([-\tau,0],H^1(g(\theta),h(\theta)))} \qquad [2.3.2]$$

for all v in $C^1([-\tau, 0], H^1(g(\theta), h(\theta)))$, where C is a positive constant.

Furthermore, if u is in X, we can deduce that u^2 is in $L^2(\Omega)$ with the estimate:

$$\|u^2\|_{L^2(\Omega)} \leq C\, \|u\|_{H^1([-\tau,0],H^1(g(\theta),h(\theta)))^2} \qquad [2.3.3]$$

for some constant $C > 0$ (independent of u). These properties ensure that the equation and boundary conditions make sense in the function space X, and provide the basis for applying the fixed-point theorem to establish the existence and uniqueness of the weak solution to the Fisher's equation.

2.3.1 Linearization :

We linearize the Fisher equation by treating the nonlinearity as a nonhomogeneous term [9] ,[5]. Define a new function $h(u) = u(1 - u)$ and consider the linear elliptic BVP: $v_t = v_{xx} + h(u)$, $g(t) < x < h(t)$, $t > 0$,

$v(t, g(t)) = 0$, $g'(t) = -\mu v_x(t, g(t))$, $t > 0$,

$v(t, h(t)) = 0$, $h'(t) = -\mu v_x(t, h(t))$, $t > 0$,

$v(\theta, x) = \varphi(\theta, x)$, $g(\theta) \leq x \leq h(\theta)$, $\theta \in [-\tau, 0]$.

Let $A(u) = v$, where v is the unique weak solution of the linearized BVP obtained in Step 2. We need to show that A is well-defined and maps X to X. To apply the Contraction Mapping Theorem, we need to show that A is a contraction on a suitable subset of X. Consider two functions $u, v \in X$. We want to estimate the difference $A(u) - A(v)$ and show that it satisfies the contraction condition.

Using the linearized BVP, let v1 and v2 be the solutions corresponding to u and v, respectively. Then we have:

$v1_t = v1_{xx} + h(u)$, $g(t) < x < h(t)$, $t > 0$,

$v1(t, g(t)) = 0$, $g'(t) = -\mu v1_x(t, g(t))$, $t > 0$,

$v1(t, h(t)) = 0$, $h'(t) = -\mu v1_x(t, h(t))$, $t > 0$,

$v1(\theta, x) = \varphi(\theta, x)$, $g(\theta) \leq x \leq h(\theta)$, $\theta \in [-\tau, 0]$,

and

$v2_t = v2_{xx} + h(v)$, $g(t) < x < h(t)$, $t > 0$,

$v2(t, g(t)) = 0$, $g'(t) = -\mu v2_x(t, g(t))$, $t > 0$,

$v2(t, h(t)) = 0$, $h'(t) = -\mu v2_x(t, h(t))$, $t > 0$,

$v2(\theta, x) = \varphi(\theta, x)$, $g(\theta) \leq x \leq h(\theta)$, $\theta \in [-\tau, 0]$.

By subtracting these equations, we obtain:

$(v1 - v2)_t = (v1 - v2)_{xx} + h(u) - h(v)$, $g(t) < x < h(t)$, $t > 0$,

$(v1 - v2)(t, g(t)) = 0$, $g'(t) = -\mu(v1 - v2)_x(t, g(t))$, $t > 0$,

$(v1 - v2)(t, h(t)) = 0$, $h'(t) = -\mu(v1 - v2)_x(t, h(t))$, $t > 0$,

$(v1 - v2)(\theta, x) = 0$, $g(\theta) \leq x \leq h(\theta)$, $\theta \in [-\tau, 0]$.

Since A is a contraction on a suitable subset of X, we can apply the Contraction Mapping Theorem. By the theorem, there exists a unique fixed point $u0 \in X$ such that $A(u0) = u0$. This fixed point corresponds to a weak solution of the original Fisher equation.

In the next part, we will look at the extension of the fixed point theorem and would focus on the generalising the solution of the Fisher's equation by the G-metric theorem and proving the existence of a weak single solution for the same which will give more the backbone to start the dynamical analysis of the system of disease propagation and population measurement.

III. G-metric space theory:

3.1 Preliminaries:

In this section, we introduce the key properties and definitions related to Gαβb-metric spaces, which are relevant to our study as defined in [12].

Definition 1: Let $X \neq \emptyset$ be a set and α, β ≥ 1 be real numbers. A function $G_b^{\alpha\beta}: X \times X \times X \rightarrow R^+$ is called a $G_b^{\alpha\beta}$-metric if it satisfies the following conditions for every x, y, z, a ∈ X:

(i) $G_b^{\alpha\beta}$(x, y, z) = 0 if x = y = z.

(ii) $G_b^{\alpha\beta}$(x, x, y) > 0 with x ≠ y.

(iii) $G_b^{\alpha\beta}$(x, x, y) ≤ $G_b^{\alpha\beta}$(x, y, z) with y ≠ z.

(iv) $G_b^{\alpha\beta}$(x, y, z) = $G_b^{\alpha\beta}$(ρ{x, y, z}), where ρ is any permutation of {x, y, z}.

(v) $G_b^{\alpha\beta}$(x, y, z) ≤ α$G_b^{\alpha\beta}$(x, a, a) + β$G_b^{\alpha\beta}$(a, y, z).

Therefore, (X, $G_b^{\alpha\beta}$) represents a $G_b^{\alpha\beta}$-metric space. When α = β, $G_b^{\alpha\beta}$becomes a G_b-metric.

Proposition 1: Let X be a $G_b^{\alpha\beta}$-metric space, and x, y, z ∈ X. The following statements hold:

(i) If $G_b^{\alpha\beta}$(x, y, z) = 0, then x = y = z.

(ii) $G_b^{\alpha\beta}$(x, y, z) ≤ α$G_b^{\alpha\beta}$(x, x, y) + β$G_b^{\alpha\beta}$(x, x, z).

(iii) $G_b^{\alpha\beta}$(x, y, y) ≤ (α + β) $G_b^{\alpha\beta}$(y, x, x).

(iv) $G_b^{\alpha\beta}$(x, y, z) ≤ α$G_b^{\alpha\beta}$(x, a, z) + β$G_b^{\alpha\beta}$(a, y, z).

Definition 2: Let X be a $G_b^{\alpha\beta}$-metric space. A sequence {x_n} is defined as:

(i) $G_b^{\alpha\beta}$-Cauchy if for each ε > 0, there exists N ∈ ℕ such that for all m, n, l ≥ N, $G_b^{\alpha\beta}$(x_n, x_m, x_l) < ε.

(ii) $G_b^{\alpha\beta}$-convergent to x⋆ ∈ X if for each ε > 0, there exists N ∈ ℕ such that for all m, n ≥ N, $G_b^{\alpha\beta}$ (x_m, x_n, x⋆) < ε.

Proposition 2: Let X be a $G_b^{\alpha\beta}$-metric space, and {x_n} be a sequence in X. The following statements are equivalent:

(i) The sequence {x_n} is $G_b^{\alpha\beta}$ −Cauchy.

(ii) For any ε > 0, there exists N ∈ ℕ such that $G_b^{\alpha\beta}$(x_n, x_m, x_m) < ε, for all m, n ≥ N.

Proposition 3: Let X be a $G_b^{\alpha\beta}$-metric space. The following statements are equivalent:

(i) {x_n} is $G_b^{\alpha\beta}$-convergent to x⋆.

(ii) $G_b^{\alpha\beta}$(x_n, x_n, x⋆) → 0 as n → ∞.

(iii) $G_b^{\alpha\beta}$(x_n, x⋆, x⋆) → 0 as n → ∞.

Definition 3: A $G_b^{\alpha\beta}$-metric space X is considered complete if every $G_b^{\alpha\beta}$-Cauchy sequence is $G_b^{\alpha\beta}$-convergent in X.

Corollary 4:

Let $(X, \preceq)$ be a partially ordered set, and suppose that there exists a G-metric ($\alpha = 1, \beta = 1$ and b=0) in X such that (X ,G) is G-complete. Consider a mapping $f : X \rightarrow X$ satisfying $f x \preceq (f x)$ for all $x \in X$. Assume the existence of nonnegative real numbers a , b , and c with a +2b +2c <1 such as given in [14],

$$G(fx, fy, fy) \leq aG(x, y, y) + b[G(x, fx, fx) + G(y, fy, fy)] + c[G(x, fy, fy) + G(y, fx, fx)] \quad [3.1.1]$$

for all comparative $x, y \in X$. Additionally, assume that X has the following property: If (x_n) is an increasing sequence that converges to x in X, then $x_n \preceq x$ for all $n \in \mathbf{N}$.

Under these conditions, f has a fixed point $u \in X$.

By understanding these properties and definitions, we lay the foundation for our subsequent analysis of fixed point results in complex-valued generalized $G_b^{\alpha\beta}$-metric spaces.

3.2 Proof:

Fisher's equation; $u_t = u_{xx} + u(1-u)$ in a integral equation [14],

$$u(x,t) = \phi(x) + \int_0^t [u_{xx} + u(1-u)]\, dt, \; t \in [0,T] \quad [3.2.1]$$

Let, $X = C([0,T])$ be the set of all continuous functions defined on [0,T], then,

$$G: X \times X \times X \rightarrow R^+$$

Then by postulates,

$$G(x,y,z) = \sup_{t\epsilon[0,T]} |x(t) - y(t)| + \sup_{t\epsilon[0,T]} |x(t) - z(t)| + \sup_{t\epsilon[0,T]} |y(t) - z(t)| \quad [3.2.2]$$

Then (X,G) is a G-complete metric space.

An ordered relation $' \preceq '$ on X by

$$x \preceq y \text{ iff } x(t) \leq y(t) \quad \forall\, t \in [0,T] \quad [3.2.3]$$

Similary, $y \leq z$ iff y(t) ≤ z(t) $\forall\ t \in [0,T]$ [3.2.4]

$x \leq z$ iff x(t) ≤ z(t) $\forall\ t \in [0,T]$ [3.2.5]

Then, (X, ≤) is a partially ordered set.

Initial hypothesis [14]:

Let, $u_{xx} + u(1-u) = k\big(t, x, u(x,t)\big)$

i) $K: [0,T] \times [0,T] \times R \to R$ *and* $g: R \to R$ *are continuous*.

ii) $t, x \in [0,T]$ and $k(t,x,u(x,t)) \leq k(t,x,\int_0^t k(t,s,u(s,t))\,ds + \phi(t)$

iii)Continuous function, $G: [0,T] \times [0,T] \to [0,+\infty]$ such that, $|k(t,x,u) - k(t,x,v)| \leq G(t,x)|u-v|$ and u,v ∈ R and each t,x ∈ $[0,T]$

iv) $\sup\limits_{t\epsilon[0,T]} \int_0^T u(t,x)\,dt \leq r$ for some r<1.

We need to show integral equation has a solution, u ∈ $C([0,T])$.

Proof : S:C([0,T]) → C([0,T]) by

$$S_u(t,x) = \int_0^T k(t,x,u(x,t))\,dt \ + \ \phi(x),\ t \in [0,T] \qquad [3.2.6]$$

Now, we can write,

$$\begin{aligned} SX(x,t) &= \int_0^T k(t,x,X(x,t))\,dt \ + \ \phi(x) \\ &\leq \int_0^T k\left(t,s,\int_0^t k(t,s,X(s,t))\,ds + \phi(t)\right) dt \ + \ \phi(x) \\ &= \int_0^T k(t,x,SX(x,t))\,dt \ + \ \phi(x) \\ &= S\big(SX(x,t)\big) \end{aligned}$$

Thus we have,

$SX \leq S(SX)$ for all x ∈ C([0,T])

For X,Y ∈ C([0,T]) with X≤Y, we have

$$\begin{aligned} G(SX,SY,SY) &= 2\sup_{t\epsilon[0,T]} |SX(x,t) - SY(x,t)| \\ &= 2\sup_{t\epsilon[0,T]} \left|\int_0^T (k(t,x,X(x,t)) - k(t,x,Y(x,t)))\,dt\right| \\ &\leq 2\sup_{t\epsilon[0,T]} \int_0^T |k\big(t,x,X(x,t)) - k(t,x,Y(x,t)\big)|\,dt \\ &\leq 2\sup_{t\epsilon[0,T]} \int_0^T G(x,t)|X(x,t) - Y(x,t)|\,dt \\ &\leq 2\sup_{t\epsilon[0,T]} |X(x,t) - Y(x,t)| \sup_{t\epsilon[0,T]} \int_0^T G(x,t)\,dt \\ &= G(X,Y,Y) \sup_{t\epsilon[0,T]} \int_0^T G(x,t)\,dt \end{aligned}$$

There is r ∈ $[0,1)$ such that,

$$\sup_{t\epsilon[0,T]} \int_0^T G(x,t)\,dt \ < \ r \qquad [3.2.7]$$

Thus G(SX,SY,SY) ≤ rG(X,Y,Y); where a=r and b,c=0 and a<1.

Therefore it satisfies the corollary 4 (3.1.1),

$G(fx,fy,fy) \ \leq \ aG(x,y,y) \ + b[G(x,fx,fx) + G(y,fy,fy)] + c[G(x,fy,fy) + G(y,fx,fx)]$

Therefore, there exists a solution of u ∈ C([0,T]) of the integral equation of fisher's equation. This is the generalised case of fisher's equation by G-metric theorem.

IV. Population Distribution Study:

The actual solution of the Fisher's equations winds up around the wave solution as u(z)=u(x-ct) for population distribution and disease propagation studies. The conditions for disease propagation can be described in [4] as follows:

Habitat Quality: The function r(ξ) represents the habitat quality, where ξ = x-ct denotes the shifting spatial domain. The function r(ξ) is continuous, nondecreasing, and bounded. It satisfies the condition r(−∞) < 0 < r(∞), indicating that there exist regions of the domain that are suitable (r(ξ) > 0) and unsuitable (r(ξ) < 0) for the growth and spread of the disease.

Shifting Habitat: The function r(x - ct) naturally divides the spatial domain into shifting regions based on the habitat quality. At any given time t, the shifting habitat edge, r(x - ct), separates the good quality habitat from the poor quality habitat. This shifting habitat edge moves with a speed c.

Persistence and Spread: The persistence and spreading dynamics of the disease depend on the speed of the shifting habitat edge, denoted as c∗(∞). If the speed of the shifting habitat edge, c > c∗(∞), the disease will die out in the entire habitat. However, if $c < c*(\infty)$, the disease will survive and spread along the shifting habitat gradient with an asymptotic spreading speed $c*(\infty)$.

Die-out Dynamics: The concept of persistence discussed above is not location-wise but refers to the virus ability to spread or move towards better resources. However, at any given location x, as time progresses, r(x - ct) becomes negative due to the condition $r(-\infty) < 0$. This leads to the extinction of the virus at that location, regardless of the overall persistence of the viruses. Therefore, understanding the point-wise "die-out dynamics" of the disease is crucial.

Traveling Wave Front: In the context of spatially diffusive population models with spatial homogeneity, traveling wave fronts (TWF) play a significant role in describing the spatial-temporal dynamics of the disease propagation. For the model system described in the passage, which includes shifting spatial heterogeneity, it is natural to consider traveling wave solutions with the shifting speed of the habitat edge.

Now we will focus on the different conditions for functions of carrying capacity and diffusion rate and their expressions in the integral solution of the above sections and will consider three different conditions of carrying capacity K and diffusion rate D in three different cases for different φ(x) from the integral equation of fisher's equation (3.2.1).

We can summarize the conditions as;

i) **When the community will flourish**:

When the carrying capacity is high and the diffusion rate is also high, it allows for efficient dispersal of individuals and ample resources to support population growth. The carrying capacity term (1-u/K) [1.1] will be smaller than 1 in this case. This condition promotes the flourishing of the community, leading to population expansion and colonization of new areas.

ii) **When the community will be stable**:

In this case, the carrying capacity and diffusion rate are at moderate levels. The carrying capacity term will cancel out the diffusion term to make u_x zero and then the system will be stable in this case. The population is in a stable equilibrium where it is neither expanding rapidly nor declining. The population size remains relatively constant, with individuals being able to adequately find resources within the available habitat.

iii) **When the community will die out**:

When the carrying capacity is low and the diffusion rate is low, the community faces limited resources and restricted dispersal. The carrying capacity of the system will be higher than the diffusion term and thus the population will die out. As a result, the population struggles to sustain itself and may eventually decline to extinction. This condition represents an unfavourable environment for the community, leading to its eventual demise.

Here in this section, we will compute the central point of the spread by the gaussian distribution methods and we considered φ(x) as $e^{(-0.5((x - \bar{x}) / \sigma)^2)}$ as a per the method of gaussian distribution. In this expression, x0 represents the center of the distribution and sigma controls the width of the distribution and repersents the standard deviation. This initial condition represents a population density that is highest at x = x0 and decreases symmetrically as we move away from x0.

By fixing the σ, D and K and initializing the range of x as x_{min}, x_{max} and num_points of x we finally compute the initial values for the x in this range.

Then we can set the range for the initial values of x and the num_points that we took to get the graphs for the different conditions using python codes and matplotlib for the graphs. Here are some examples for the different conditions and different initial values that we took to get varying results;

Case 1 (low D and low K, population will die out) :

a) σ = 1.0 , D = [0.1, 0.5, 1.0] , K = [100, 500, 1000], x_{min} = -10.0, x_{max} = 10.0, num_points = 1000 and φ(x) = $e^{(-0.5((x - \bar{x}) / \sigma)^2)}$.

In the range of x from [10, -10], initial values which have the maximum values are

u(-0.05) = 999.198719

u(-0.03) = 999.799619

u(-0.01) = 1000.000000

u(0.01) = 999.799619

u(0.03) = 999.198719

u(0.05) = 998.198020

and, Central Location (x0): -0.010010010010010006.

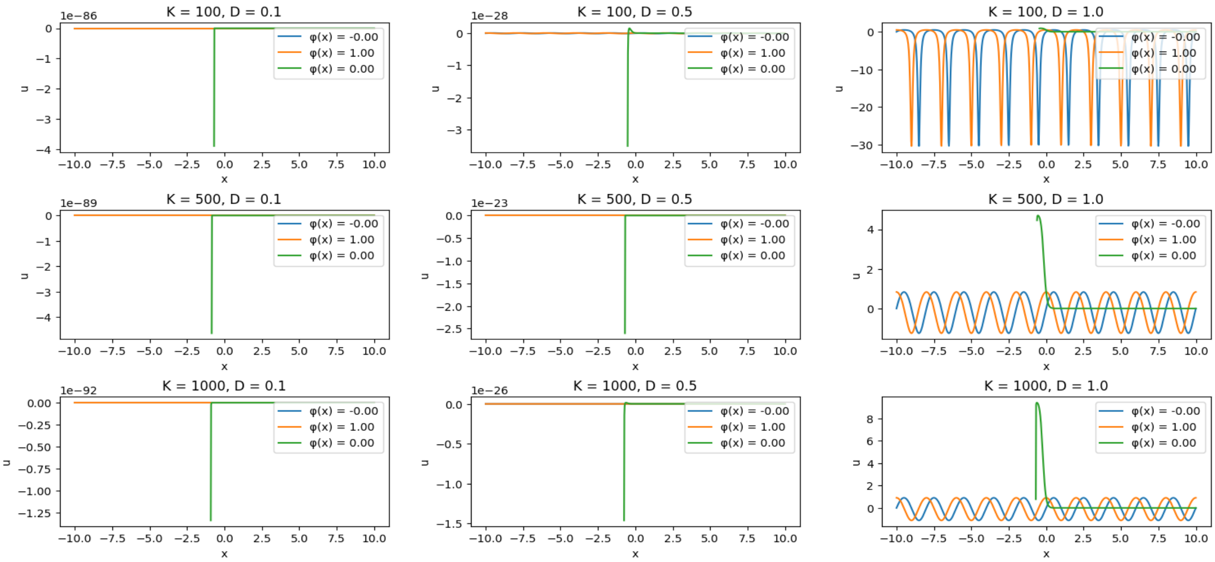

Fig 1. In this case of low D and low K, the system has in the first two cases the expansion died out easily and in the last cases it should a continuous behaviour

b) σ = 1.0, K = [10, 50, 100], D = [0.01, 0.05, 1.0], num_points = 100, x = [-15, 15, num_points] and φ(x) = $e^{(-0.5((x-\bar{x})/\sigma)^2)}$.

In the range of x from [15, -15], initial values which have the maximum values are
u(-0.76) = 83.222310
u(-0.45) = 95.512440
u(-0.15) = 100.000000
u(0.15) = 95.512440
u(0.45) = 83.222310
u(0.76) = 66.151466

and, Central Location (x0): -0.1515151515151505.

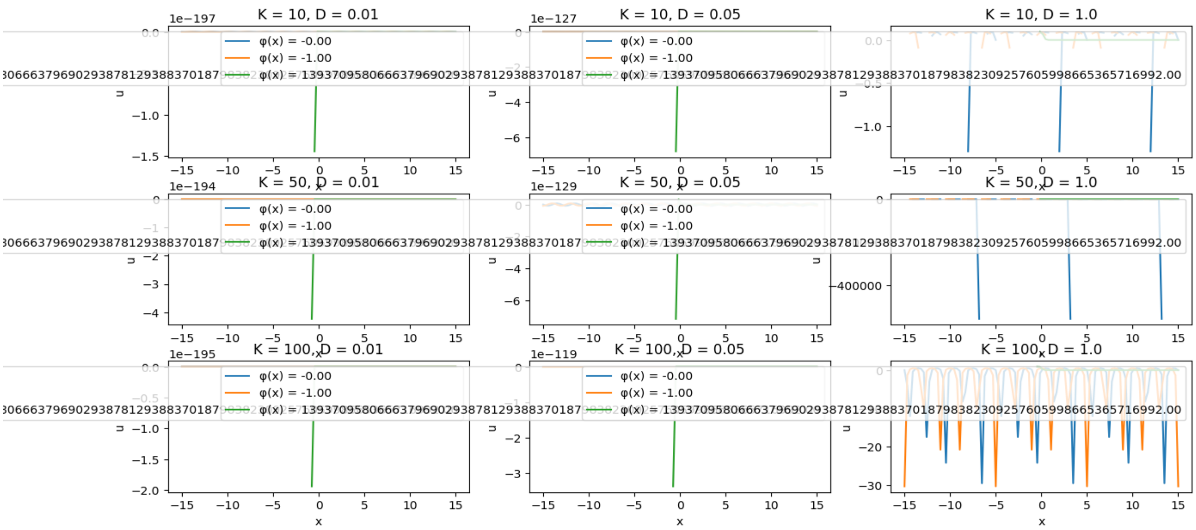

Fig 2: In this case with decrease in range of K and D and low num_points the disruption becomes less and we can find that the population die out faster

c) σ = 1.0, K = [10, 50, 100], D = [0.01, 0.05, 1.0], num_points = 10, x = [-0.5, 0.5, num_points] and φ(x) = $e^{(-0.5((x-\bar{x})/\sigma)^2)}$.

In the range of x from [0.5, -0.5], initial values which have the maximum values are

u(-0.28) = 97.561098
u(-0.17) = 99.384617
u(-0.06) = 100.000000
u(0.06) = 99.384617
u(0.17) = 97.561098
u(0.28) = 94.595947

and, Central Location (x0): -0.05555555555555558.

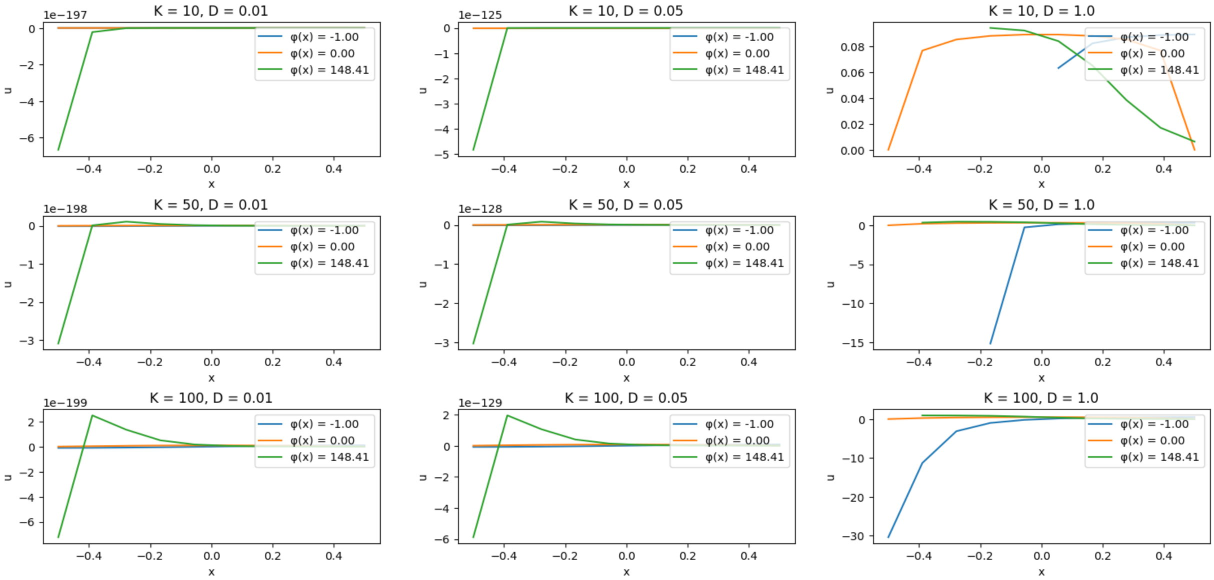

Fig 3: In this case with more decrease in range D and low num_points the disruption from the original graph becomes less and we can find that the population die out more faster, but with low K and high D in the three cases shows more chaotic nature of the population which settles down to stability.

As a result, from fig1, fig2, fig3 and in these three cases of the low D and low K shows the decrease in the fluctuation in the three different parts with changing K and D ranges and ultimately reaches stability and population die out.

Case 2(Moderate D and Moderate K, population will be stable): σ = 1000.0 , D = [0.5,0.7,1.0] , K = [800, 1000, 1200] , x_min = -1.0, x_max = 1.0, num_points = 500 and φ(x) = $e^{(-0.5((x-\bar{x})/\sigma)^2)}$.

In the range of x from [1.0, -1.0], initial values which have the maximum values are
u(-0.05) = 799.999999
u(-0.04) = 799.999999
u(-0.03) = 800.000000
u(-0.02) = 800.000000
u(-0.01) = 800.000000
u(-0.00) = 800.000000
u(0.00) = 800.000000
u(0.01) = 800.000000
u(0.02) = 800.000000
u(0.03) = 799.999999
u(0.04) = 799.999999
u(0.05) = 799.999999
and, Central Location (x0): -0.002004008016032177.

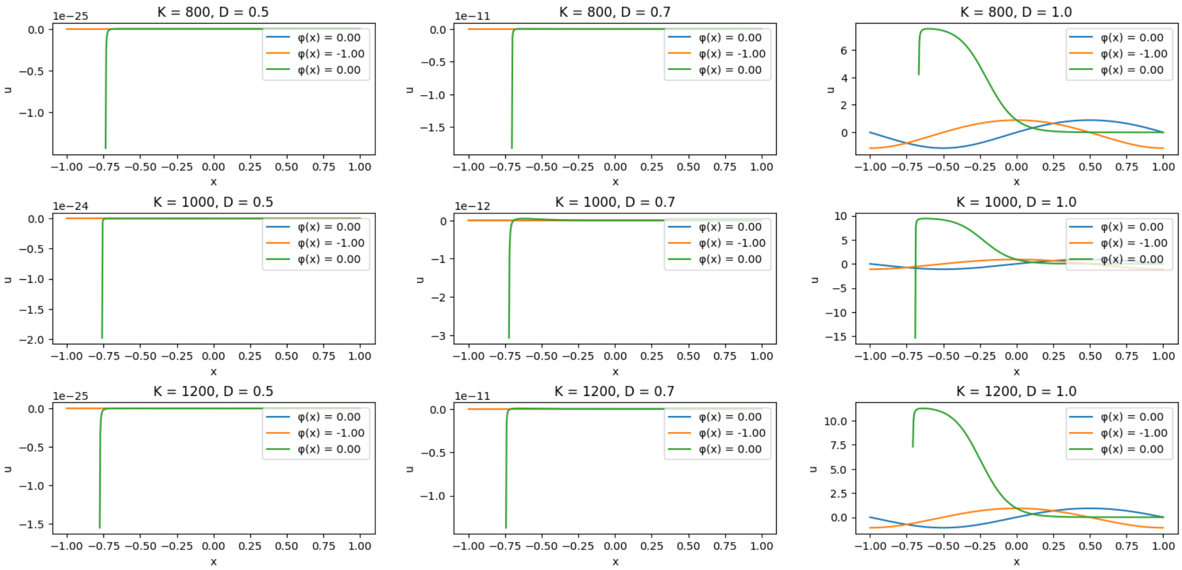

Fig 4: In this case, with moderate K and D we find that the fluctuation is more than the first part and the population reaches a more stable point.

From the fig 4, we can conclude that the population fluctuation is more initially and ultimately it reaches stability but in the last cases with high K and high D the fluctuation is more than original.

Case 3(high D and high K, population will flourish):

a) σ = 100000 ,K = [1000, 2000, 4000] ,D = [0.8, 0.9, 1.0] ,num_points = 100, x = [-0.8, 0.8, num_points] and $\varphi(x) = e^{(-0.5((x-\bar{x})/\sigma)^2)}$.

In the range of x from [0.8, -0.8], initial values which have the maximum values are

u(-0.04) = 4000.000000

u(-0.02) = 4000.000000

u(-0.01) = 4000.000000

u(0.01) = 4000.000000

u(0.02) = 4000.000000

u(0.04) = 4000.000000

and, Central Location (x0): -0.008080808080808133.

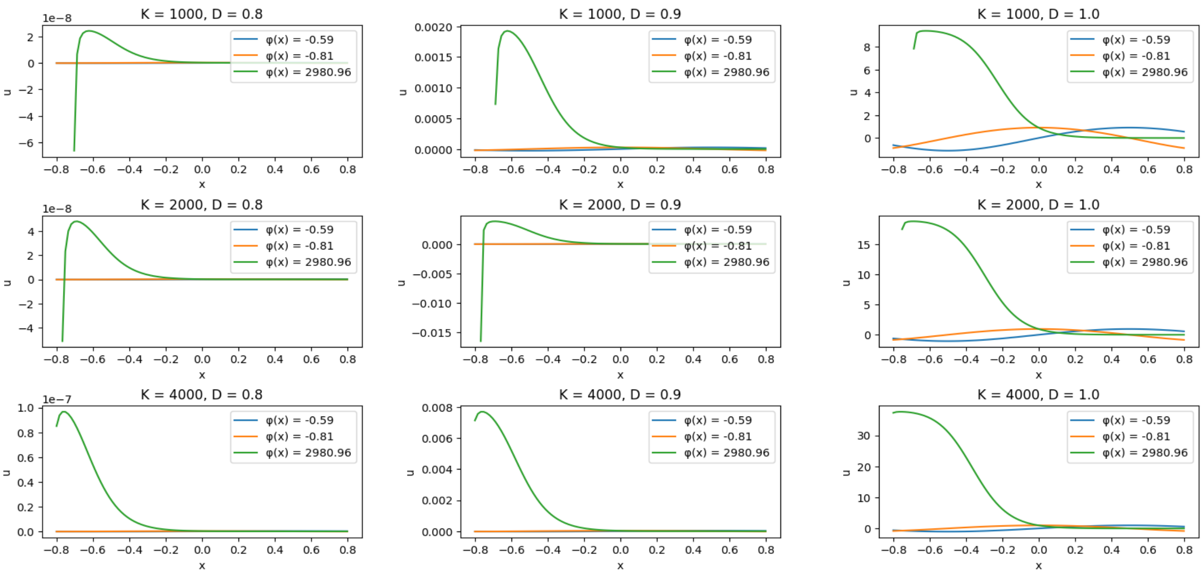

Fig 5: In this case, with increase in σ (gaussian distribution) and ranges in K and D the density fluctuation is high and initially the population increases exponentially

b) σ = 100000, K = [1000, 2000, 4000] ,D = [0.8, 0.9, 1.0] ,num_points = 100 ,x = [-15, 15, num_points] and φ(x) = $e^{(-0.5((x-\bar{x})/\sigma)^2)}$.

In the range of x from [15, -15], initial values which have the maximum values are
u(-0.76) = 4000.000000
u(-0.45) = 4000.000000
u(-0.15) = 4000.000000
u(0.15) = 4000.000000
u(0.45) = 4000.000000
u(0.76) = 4000.000000

and, Central Location (x0): -0.1515151515151505.

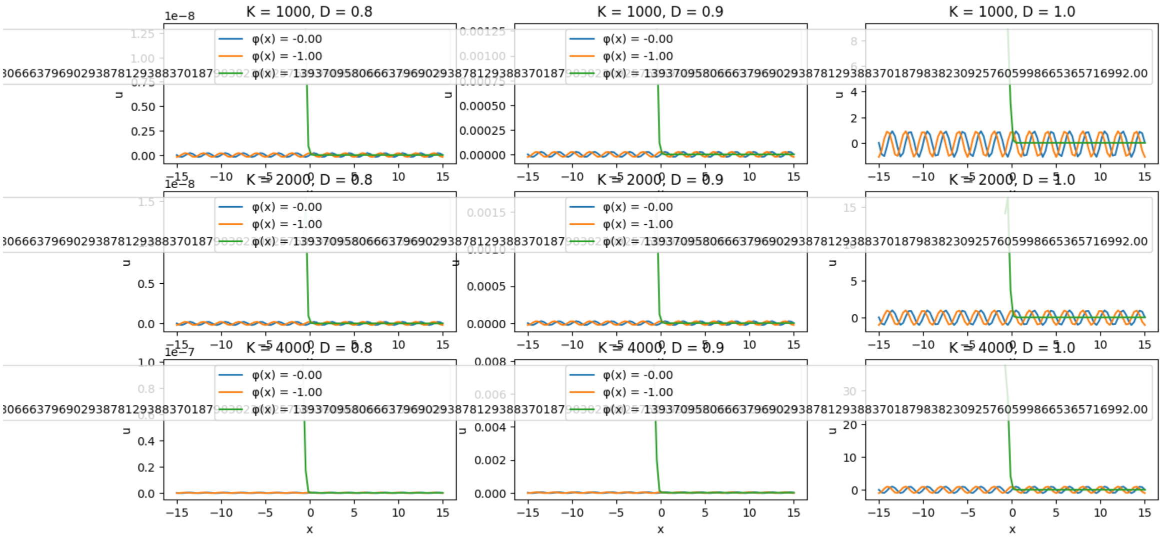

Fig 6: In this case, with increase in range of x the density fluctuation is observably high and the population increases exponentially and the system becomes more chaotic

From Fig 5 and Fig 6, we can conclude that with high gaussian distribution σ and high ranges of K and D create high fluctuation rate and the population increases gradually.

V. Morse Theorem:

Let's consider using Morse theory to compute critical points for the Fisher's equation to find the stability of the critical points and to determine the dynamical analysis of the system referred from [7],[13],[1].
$u_t = u_{xx} + u(1 - u)$, $g(t) < x < h(t)$, $t > 0$, in Ω
$u(t, g(t)) = 0$, $g(t) = -\mu u_x(t, g(t))$, $t > 0$, on ∂Ω
$u(t, h(t)) = 0$, $h(t) = -\mu u_x(t, h(t))$, $t > 0$, on ∂Ω
$u(\theta, x) = \varphi(\theta, x)$, $\theta \in [-\tau, 0]$, $x \in X$ on ∂Ω
where Ω is a bounded domain in R^3 with smooth boundary ∂Ω.
To apply Morse theory, we need to rewrite the fisher equations as a variational problem.
To find a variational formulation for the Fisher's equations, we can introduce an auxiliary variable v = u(1 - u) . Then, the original system can be rewritten as:
$u_t = u_{xx} + v$, in Ω
where, v = u(1 - u), in Ω
$u(t, g(t)) = 0$, $g'(t) = -\mu u_x(t, g(t))$, $t > 0$, on ∂Ω
$u(t, h(t)) = 0$, $h'(t) = -\mu u_x(t, h(t))$, $t > 0$, on ∂Ω
$u(\theta, x) = \varphi(\theta, x)$, $\theta \in [-\tau, 0]$, $x \in X$, on ∂Ω
Next, we define the following functionals [1], considering the functional J(u, v) defined as:

$$J(u, v) = \int_\Omega (|\nabla u|^2 + v^2)\,dx \quad [5.1]$$

$$K(u, v) = \int_\Omega (u(1 - u) - v)\,dx \quad [5.2]$$

We can now express the original problem as a variational problem by seeking critical points of the functional J subject to the constraint given by K:

$\delta J(u, v; w, z) = 0$ for all (w, z) satisfying $\delta K(u, v; w, z) = 0$,
where $\delta J(u, v; w, z)$ and $\delta K(u, v; w, z)$ represent the first variations of the functionals J and K, respectively.
To compute the Hessian matrix associated with the variational problem derived from the Fisher's equations, we need to take the second variations of the functionals J and K. Below are the Hessian matrix:
For the functional J(u, v):

$$H_J = \left| \frac{\delta^2 J(u,v)}{\delta u^2} \right| + 2 \left| \frac{\delta^2 J(u,v)}{\delta u \delta v} \right| + \left| \frac{\delta^2 J(u,v)}{\delta v^2} \right| \qquad [5.3]$$

For the functional K(u, v):

$$H_K = \left| \frac{\delta^2 K(u,v)}{\delta u^2} \right| + 2 \left| \frac{\delta^2 K(u,v)}{\delta u \delta v} \right| + \left| \frac{\delta^2 K(u,v)}{\delta v^2} \right| \qquad [5.4]$$

Morse theory introduces the concept of the Morse index, which is the number of negative eigenvalues of the Hessian matrix associated with a critical point. The Morse index helps classify the critical points and understand their stability properties.

5.1 Dynamical System analysis by Morse Index:

The Morse index is a numerical quantity that helps in characterizing the stability and behaviour of critical points in a dynamical system. To compute the Morse index, we first need to obtain the Hessian matrix associated with the system. The Hessian matrix captures the local behaviour of the system near a critical point and provides information about the curvature of the solution manifold. In the context of the Fisher's equation, the Hessian matrix is constructed from the second variations of the functionals J and K, which are derived from the variational formulation of the original system.

Once we have the Hessian matrix, we can compute its eigenvalues. The eigenvalues represent the characteristic values of the matrix and give insights into the stability of the critical point. A negative eigenvalue indicates instability, while a positive eigenvalue suggests stability. The Morse index corresponds to the number of negative eigenvalues associated with a critical point [5].

By examining the eigenvalues of the Hessian matrix, we can determine the stability and type of critical points by coding with and the graphs by matplotlib. For example, if all eigenvalues are negative, the critical point is stable. If there are positive eigenvalues, it indicates instability or the presence of more complex dynamics such as bifurcations or limit cycles.

The Morse index provides a quantitative measure of the stability and complexity of critical points. A higher Morse index implies greater instability and the potential for more intricate dynamics. It helps in classifying the critical points and understanding the qualitative behaviour of the system.

Case 1: (low D and low K, population will die out)

N = 130 ,L = 30 ,dx = L / (N - 1);
where N is number of grid points, L is length of the domain and dx is the grid spacing.
x = [-30, L, N], u = [-0.997, 0.997, 130], v = [-1.991, 0.002, 130]
Morse Index (J): 32
Morse Index (K): 66
Average Critical Points (J): -5.773159728050814e-16
Average Critical Points (K): -5.773159728050814e-16
Average Eigenvalues (J): -23.9932760541779
Average Eigenvalues (K): -1.0000000000000004
Hessian Matrix (J):

```
[[17.83175597   0.          0.         ...  0.          0.          0.        ]
 [ 0.           9.56101986  0.         ...  0.          0.          0.        ]
 [ 0.           0.         -0.39574021 ...  0.          0.          0.        ]
 ...
 [ 0.           0.          0.         ... 15.49303205  0.          0.        ]
 [ 0.           0.          0.         ...  0.         82.19792536  0.        ]
 [ 0.           0.          0.         ...  0.          0.         83.3075002 ]]
```

Hessian Matrix (K):

```
[[-50.10680818   0.           0.          ...  0.          0.          0.        ]
 [  0.          -55.47767913  0.          ...  0.          0.          0.        ]
 [  0.           0.         -12.91657919  ...  0.          0.          0.        ]
 ...
 [  0.           0.           0.          ... 10.91657919  0.          0.        ]
 [  0.           0.           0.          ...  0.         53.47767913  0.        ]
 [  0.           0.           0.          ...  0.          0.         48.10680818]]
```

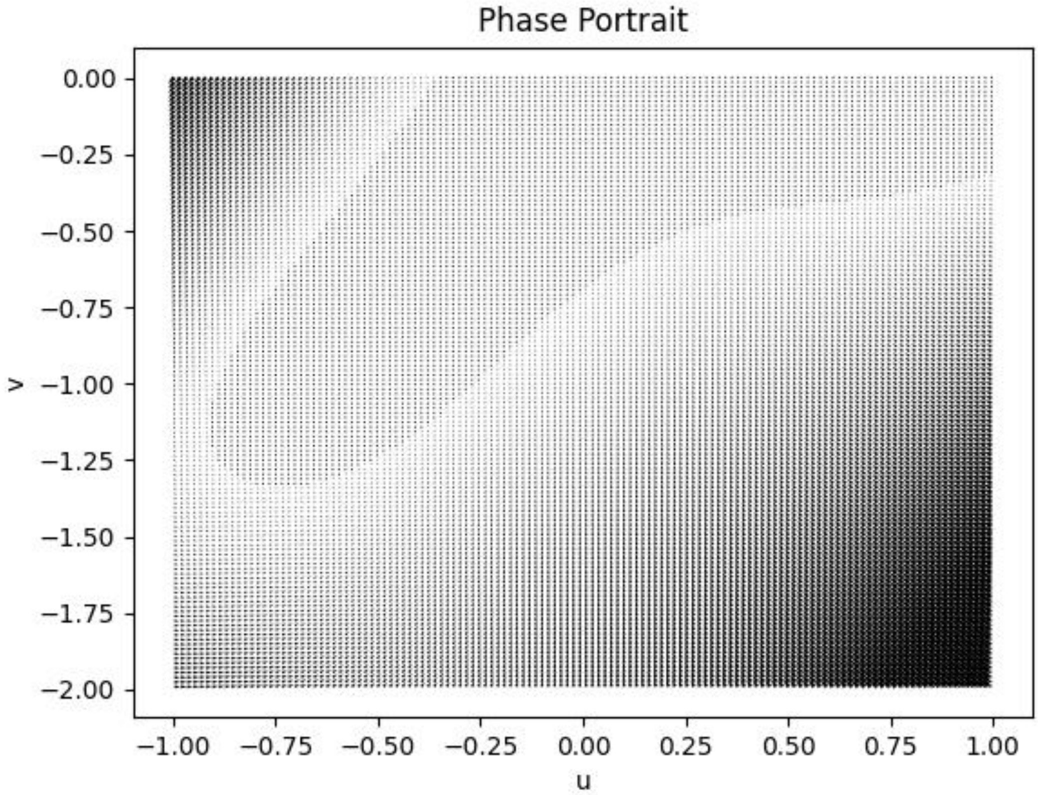

Fig 7: In this case, as the morse index are low, so there is more stability and ultimately the population dies out, as the gradient arrows mostly cancels each other and obtains stability

Case 2 (Moderate D and Moderate K, population will be stable):
N = 130 ,L = 15 ,dx = L / (N - 1)
x = [-15, L, N]
u = [-0.732, 0.732, 130]
v = [-1.267, 0.196, 130]

Morse Index (J): 41
Morse Index (K): 66

Average Critical Points (J): -4.987463433700703e-16
Average Critical Points (K): -4.987463433700703e-16

Average Eigenvalues (J): 0.9999999999988893
Average Eigenvalues (K): -0.9999999999999991

Hessian Matrix (J):

```
[[ 46.78478259   0.          0.        ...  0.          0.    0.        ]
 [  0.         100.3853488   0.        ...  0.          0.    0.        ]
 [  0.           0.        166.6150004 ...  0.          0.    0.        ]
 ...
 [  0.           0.          0.        ... 35.66351193   0.    0.        ]
 [  0.           0.          0.        ...  0.   12.47156019   0.        ]
 [  0.           0.          0.        ...  0.          0.   -3.60219026]]
```

Hessian Matrix (K):

```
[[ 36.79022964   0.          0.        ...  0.          0.    0.        ]
 [  0.          64.93534146  0.        ...  0.          0.    0.        ]
```

```
[ 0.          0.          97.21361635 ...   0.          0.    0.        ]
...
[ 0.          0.          0.          ... -99.21361635   0.    0.        ]
[ 0.          0.          0.          ...   0.  -66.93534146    0.        ]
[ 0.          0.          0.          ...   0.          0.    -38.79022964]]
```

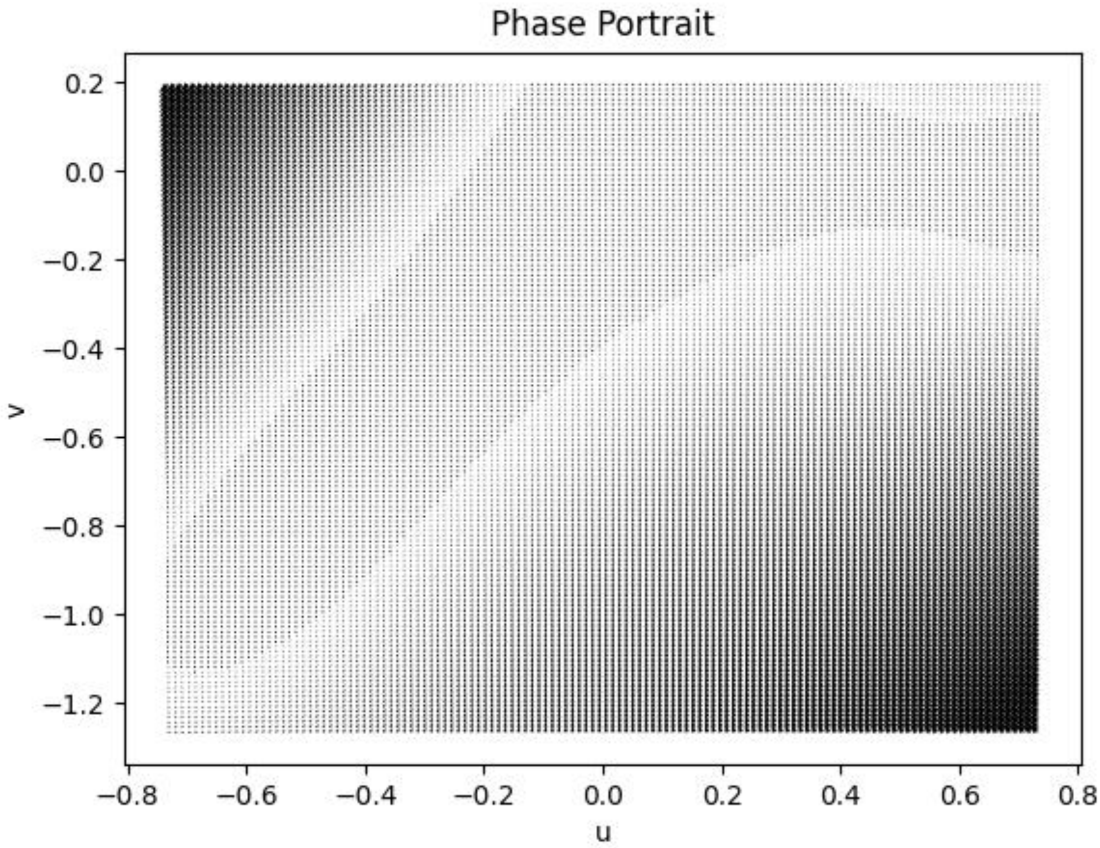

Fig 8: In this case, as the morse index are more than earlier so there is more instability but the population will be stable, as the gradient arrows are all stabilised

Case 3(high D and high K, population will flourish):
N = 130 ,L = 0.1 ,dx = L / (N - 1)
x = [-0.1, L, N]
u = [-0.005, 0.005, 130]
v = [-0.005, -0.004, 130]
Morse Index (J): 130
Morse Index (K): 67

Average Critical Points (J): -4.099285014000578e-17
Average Critical Points (K): -4.099285014000578e-17

Average Eigenvalues (J): 60.93980219700967
Average Eigenvalues (K): 0.9999999999978435

Hessian Matrix (J):

```
[[ -8.13006752   0.          0.          ...   0.          0.    0.        ]
[ 0.        -12.48914282   0.          ...   0.          0.    0.        ]
[ 0.          0.        -17.4350454   ...   0.          0.    0.        ]
...
```

```
[ 0.        0.        0.       ... -64.76738039   0.    0.       ]
[ 0.        0.        0.       ...  0.  -48.35521359    0.       ]
[ 0.        0.        0.       ...  0.        0.    -32.16298436]]
```

Hessian Matrix (K):

```
[[ 17.02468763   0.         0.        ...  0.         0.     0.       ]
[ 0.        25.89955307   0.        ...  0.         0.     0.       ]
[ 0.         0.        34.49925125 ...  0.         0.     0.       ]
...
[ 0.         0.         0.        ... -36.49925125   0.     0.       ]
[ 0.         0.         0.        ...  0.    -27.89955307   0.       ]
[ 0.         0.         0.        ...  0.         0.    -19.02468763]]
```

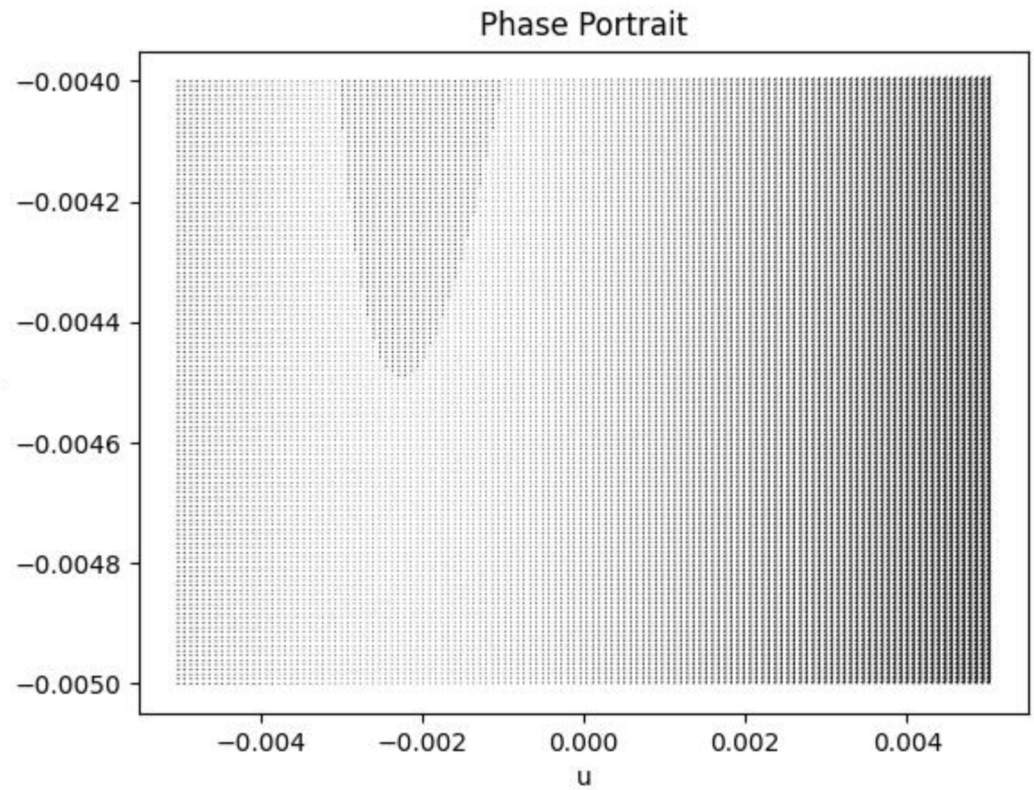

Fig 9: In this case, as the morse index are highest so there is more instability and the population will be unstable and increase gradually as the gradient arrows are pointing in the same direction and rarely gets cancelled

As a result in the Case 1, the eigenvalues were negative representing the stability in the system and with low morse index the population dies out, in Case 2, the eigenvalues were close to zero and with higher morse index indicating the critical stability the population went and finally in Case 3, the eigenvalues were highly positive suggesting the instability and chaotic nature, and the high morse index indicates that the population increases exponentially and become unstable.

VI. Conclusion:

In this project, we utilized the G-metric and Fixed point theorem to derive a single weak solution for the Fisher's equation with diffusion. We then applied Morse theory to calculate the Morse index, which provided insights into the system's stability and complexity. By analysing the eigenvalues and plotting a phase portrait, we visualized the system's dynamics. Additionally, the trajectories analysis graphs illustrated the relationship between the carrying capacity (K) and diffusion rate (D), highlighting critical points and transitions in the system. Overall, our approach allowed us to understand the system's behaviour and make connections between its parameters and dynamics.

Acknowledgement:

I would like to acknowledge the valuable guidance and support provided by Professor Binod Chandra Tripathy throughout the duration of this project. His expertise and feedback have been instrumental in shaping the direction and execution of my work.

Conflict of Interest:

There is no conflict of interest to declare regarding this project.

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