International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 11 Issue 1 || Jan. - Feb., 2023 || PP 21-27

Quasi mean value theorem for plane triangles

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Abstract: *The existing middle line theorem gives the square relation between three hypotenuses and the middle line. In this paper, the linear relation between three hypotenuse and midline in a plane triangle is proved by analytic geometry. Firstly, directly from the original research ideas of multi - station passive location theory, the median relation among three radial distances of one-dimensional double-basis symmetric array is directly transformed into a plane triangle based on the extended results of linear solution of the one-dimensional double-basis symmetric arrays, and the quasi mean value theorem between midline and hypotenuse on either side of the midline is given. Then, the process of proving the quasi-mean value theorem is described again in terms of pure plane geometry.*

Key words: *Speculative geometry; planimetry; analytic geometry; mean value theorem; oblique triangle; midline; the midline theorem; Pappus Law; passive location; double-base array*

Date of Submission: 28-01-2023 Date of Acceptance: 10-02-2023

I. Introduction

There are many famous theorems about plane geometry. Among them, the middle line theorem (Pappus Law) gives the square relation between the middle line and the three hypotenuses: the sum of the squares of the hypotenuses on both sides of the middle line is equal to twice the sum of squared of half of the bottom side and the square of the middle line.

One of the author's contributions to the theory of multi-station passive location is to study and give the analytical method of multi-station path differential location equation [1]. Furthermore, many interesting intrinsic properties of one-dimensional double - base symmetric arrays are revealed. One of the properties is the median relationship among the three radial distances [2]. This paper deduces a pure mathematical geometry theorem based on the result of the author's engineering theory research.

II. Proof method based on passive location theory

So far, most of the researches on passive positioning system only arrange mathematical equations from the angle of solving unknown quantities, but do not deeply study the internal correlation between various parameters. In fact, one-dimensional linear arrays do have many interesting inherent properties.

The authors' existing research results show that based on the linear solution of one dimensional double-base symmetric array, using the relation between adjacent path difference, the median relationship among three radial distances of the three detection points in one-dimensional double-base symmetric array can be given by substituting variables for one dimensional double - base direction - finding solutions while proving that there are properties similar to arithmetic series between adjacent path differences.

2.1 Linear solutions of one-dimensional two-basis array

For the one-dimensional double-base isometric linear array shown in Fig. 1, according to the passive positioning technique, two path differences between two adjacent baselines can be obtained by detection:

$$\Delta r_{12} = r_1 - r_2 \qquad (1)$$

$$\Delta r_{23} = r_2 - r_3 \tag{2}$$

Where, r_i is the radial distance $(i = 1-3)$; $\Delta r_{j(j+1)}$ the path difference $(j = 1,2)$.

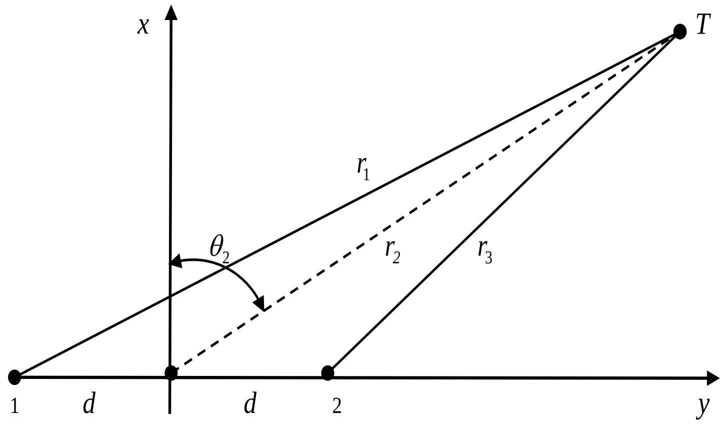

Figure 1

If the midpoint of the whole array is taken as the origin of coordinates, the following two auxiliary geometric equations can be derived by the law of cosines:

$$\begin{aligned} r_1^2 &= r_2^2 + d^2 - 2r_2 d\cos\left(90^0 + \theta_2\right) \\ &= r_2^2 + d^2 + 2r_2 d\sin\theta_2 \end{aligned} \tag{3}$$

$$\begin{aligned} r_3^2 &= r_2^2 + d^2 - 2r_2 d\cos\left(90^0 - \theta_2\right) \\ &= r_2^2 + d^2 - 2r_2 d\sin\theta_2 \end{aligned} \tag{4}$$

Where: d is the length of baseline; θ_2 the arrival angle at the midpoint of the array.

Due to: $x = r_2 \sin\theta_2$, the geometric auxiliary equation can be rewritten as:

$$r_1^2 = r_2^2 + d^2 + 2d \cdot x \tag{5}$$

$$r_3^2 = r_2^2 + d^2 - 2d \cdot x \tag{6}$$

Where: x is the abscissa of the rectangular coordinate system.

At this point, if the path differences (1) and (2) of the two adjacent baselines are substituted into geometric auxiliary equations (5) and (6), a system of linear equations of two variables containing unknown variables x and r_2 can be obtained after transposition:

$$2d \cdot x - 2\Delta r_{12} r_2 = -d^2 + \Delta r_{12}^2 \tag{7}$$

$$2d \cdot x - 2\Delta r_{23} r_2 = d^2 - \Delta r_{23}^2 \tag{8}$$

From this, the angle of arrival of the target can be solved directly:

$$\sin\theta_2 = \frac{\left(d^2 - \Delta r_{12}^2\right)\Delta r_{23} + \left(d^2 - \Delta r_{23}^2\right)\Delta r_{12}}{d\left(2d^2 - \Delta r_{12}^2 - \Delta r_{23}^2\right)} \tag{9}$$

2.2 Single-base midpoint direction finding formula

The high-order term of path difference in one-dimensional double-basis direction finding formula (9) is approximated as follows: $\Delta r_{12} \approx \Delta r_{23}$. After simplifying accordingly:

$$\sin\theta_2 = \frac{\left(d^2 - \Delta r_{12}^2\right)\left(\Delta r_{12} + \Delta r_{23}\right)}{2d\left(d^2 - \Delta r_{12}^2\right)} \approx \frac{\left(\Delta r_{12} + \Delta r_{23}\right)}{2d} \tag{10}$$

Due to :

$$\Delta r_{13} = r_1 - r_3 = \left(r_1 - r_2\right) + \left(r_2 - r_3\right) = \Delta r_{12} + \Delta r_{23} \tag{11}$$

Where, Δr_{13} is the path difference corresponding to the total length of array baseline.

The single basis direction finding solution is obtained

$$\sin\theta_2 \approx \frac{\Delta r_{13}}{2d} \tag{12}$$

It should be noted here that the reference datum for single base direction finding is at the midpoint of the single baseline, not at the left and right endpoints of the single baseline.

2.3 Approximate solution of the median relation

For geometric auxiliary equations (3) and (4) obtained by the law of cosines, after subtracting the two equations, we get:

$$r_1^2 - r_3^2 = 4r_2 d\sin\theta_2 \tag{13}$$

The left-hand side of the equation is developed according to the square deviation formula:

$$\left(r_1 + r_3\right)\left(r_1 - r_3\right) = r_1^2 - r_3^2$$

The right hand side of the equation uses the single basis midpoint direction finding solution (12)

$$2d\sin\theta_2 = \Delta r_{13} = r_1 - r_3$$

The approximate median relation among the three radial distances can be obtained

$$r_1 + r_3 \approx 2r_2 \tag{14}$$

Since the single basis midpoint direction finding solution is an approximate solution, the median relation of radial distance among the three stations is approximate.

2.4 Strict Derivation

Using the equation (11) between the path difference corresponding to two adjacent baselines and the path difference corresponding to the total length of the array baselines, replace the path difference Δr_{12} or Δr_{23} displacement in one dimensional double basis path difference direction finding formula (9), and get, respectively:

$$\sin\theta_2 = \frac{\left(d^2 - \Delta r_{12}^2\right)\left(\Delta r_{13} - \Delta r_{12}\right) + \left[d^2 - \left(\Delta r_{13} - \Delta r_{12}\right)^2\right]\Delta r_{12}}{d\left[2d^2 - \Delta r_{12}^2 - \left(\Delta r_{13} - \Delta r_{12}\right)^2\right]} \tag{15}$$

$$\sin\theta_2 = \frac{\left[d^2 - \left(\Delta r_{13} - \Delta r_{23}\right)^2\right]\Delta r_{23} + \left(d^2 - \Delta r_{23}^2\right)\left(\Delta r_{13} - \Delta r_{23}\right)}{d\left[2d^2 - \left(\Delta r_{13} - \Delta r_{23}\right)^2 - \Delta r_{23}^2\right]} \tag{16}$$

Take adjacent path differences Δr_{12} and Δr_{23} as unknowns, from which we can solve:

$$\Delta r_{12} = r_1 - r_2 = 0.5\Delta r_{13} + \Delta a \tag{17}$$

$$\Delta r_{23} = r_2 - r_3 = 0.5\Delta r_{13} - \Delta a \tag{18}$$

Thereinto, Δa is:

$$\Delta a = 0.5\sqrt{\frac{\Delta r_{13}^2(r_{13} + 2d\sin\theta_2) - 4d(\Delta r_{13}^2\sin\theta_2 + d\cdot r_{13} - 2d^2\sin\theta_2)}{r_{13} + 2d\sin\theta_2}} \tag{19}$$

Obviously, according to the mathematical expression of equations (17) and (18), the path difference of the double basis can be written in the form of arithmetic series, i.e

$$\Delta r_{j(j+1)} = 0.5\Delta r_{13} + 2(1.5 - j)\Delta a \quad (j = 1,2) \tag{20}$$

Therefore, from the point of view of mathematical analysis, $2\Delta a$ (or Δa) is the tolerance of arithmetic series. If the two equations are subtracted based on the path difference on the right side of Equation (17) and Equation (18), it can be obtained

$$2\Delta a = \Delta r_{12} - \Delta r_{23} \tag{21}$$

Therefore, from the point of view of plane geometry, $2\Delta a$ is actually to represent the difference between the adjacent path differences.

If the two equations are subtracted based on the radial distance in the middle of Equations (17) and (18), it can be obtained

$$\frac{r_1 + r_3}{2} = r_2 + \Delta a \tag{22}$$

That is, the arithmetic mean value of the radial distance between the left and right stations is the sum of the radial distance of the midpoint and the tolerance.

If the tolerance Δa of the arithmetic series between the adjacent path differences is a very small amount and can be ignored, the approximate median relation (14) among three radial distances of the three stations can be obtained directly.

2.5 Geometric Deduction

As shown in Figure 2, let the midpoint of side BC of triangle ΔABC be P, and Δl_1 is the difference between the midline and the hypotenuse on one side of the midline

$$\Delta l_1 = AB - AP \tag{23}$$

Δl_2 is the difference between the midline and the hypotenuse corresponding to the other side of the midline

$$\Delta l_2 = AP - AC \tag{24}$$

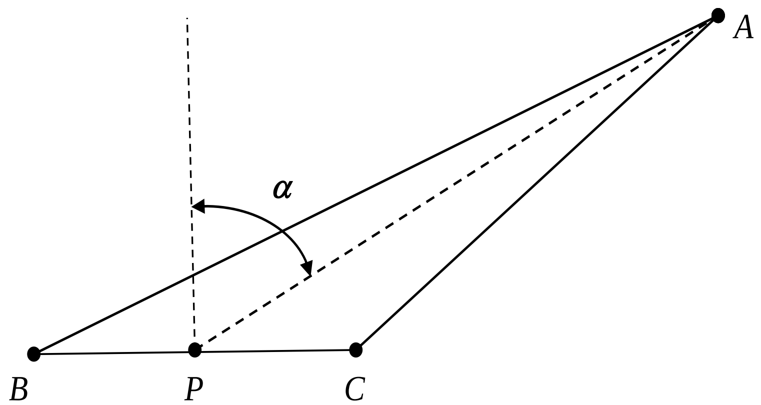

Figure 2

According to the above deduction result based on the positioning solution of one-dimensional double-basis symmetric array, Equation (22), should be

$$\frac{AB+AC}{2}=AP+\Delta A \tag{25}$$

Where, after directly using the result of Equation (21), can be

$$\Delta A=0.5[(AB-AP)-(AP-AC)] \tag{26}$$

Thus, a quasi-mean value theorem based on plane geometry is obtained: the sum of half of the two hypotenuses of the opposite sides of the median line of a triangle is equal to sum of the midline and differences of two adjacent differences between the hypotenuses and midline.

According to Equation (19), the difference of the difference value between two hypotenuses and the midline in a triangle can also be calculated by the formula

$$\Delta A=0.5\sqrt{\frac{\Delta l^2(\Delta l+BC\sin\alpha)-2BC(\Delta l^2\sin\alpha+0.5BC\cdot\Delta l-0.5BC^2\sin\alpha)}{\Delta l+BC\sin\alpha}} \tag{27}$$

Where, $\Delta l=AB-AC$ is the difference of the hypotenuse corresponding to the two sides of the midline, and α is the included angle between the perpendicular line at the midpoint of the base and the midline.

If the result of equation (14) is directly used, then

$$\frac{AB+AC}{2}\approx AP \tag{28}$$

That is: half of the sum of the hypotenuse corresponding to the two sides of the midline of a triangle is approximately equal to the length of the middle line.

III. Pure geometric proof

III.1 The angle between the midline and the perpendicular at the midpoint of the bottom

For triangle ΔABC as shown in Figure 2, let the midpoint of base BC be P, and Δl_1 is the difference between the midline and the hypotenuse on one side of the midline

$$\Delta l_1=AB-AP \tag{23}$$

Δl_2 is the difference between the midline and the hypotenuse corresponding to the other side of the midline:

$$\Delta l_2=AP-AC \tag{24}$$

The following two auxiliary geometric equations can be derived from the law of cosines:

$$\begin{aligned}AB^2&=AP^2+BP^2-2AP\cdot BP\cos\left(90^0+\alpha\right)\\&=AP^2+BP^2+2AP\cdot BP\sin\alpha\end{aligned} \tag{29}$$

$$\begin{aligned}AC^2&=AP^2+PC^2-2AP\cdot PC\cos\left(90^0-\alpha\right)\\&=AP^2+BP^2-2AP\cdot BP\sin\alpha\end{aligned} \tag{30}$$

Where, $BP=PC$ is half of the bottom length; α is the included angle between the perpendicular line at the midpoint of the base and the midline.

Let the midpoint of the base be the origin of the Cartesian coordinate system. Since $x=AP\sin\alpha$, the geometric auxiliary equation can be rewritten as

$$AB^2 = AP^2 + BP^2 + 2BP \cdot x \tag{31}$$

$$AC^2 = AP^2 + BP^2 - 2BP \cdot x \tag{32}$$

At this point, if the two adjacent difference values (1) and (2) between the two hypotenuses and the midline are substituted into geometric auxiliary equations (5) and (6), a system of linear equations of two variables containing unknown variables x and AP can be obtained after transposition

$$2BP \cdot x - 2\Delta l_1 AP = -BP^2 + \Delta l_1^2 \tag{33}$$

$$2BP \cdot x - 2\Delta l_2 AP = BP^2 - \Delta l_2^2 \tag{34}$$

From this we can directly solve the angle α between the midline and the perpendicular line at the midpoint of the base side:

$$\sin\alpha = \frac{\left(BP^2 - \Delta l_1^2\right)\Delta l_2 + \left(BP^2 - \Delta l_2^2\right)\Delta l_1}{BP\left(2BP^2 - \Delta l_1^2 - \Delta l_2^2\right)} \tag{35}$$

III.2 **Differential relationship between the hypotenuse and the midline**

Using relations:

$$\Delta l = AB - AC = (AB - AP) + (AP - AC) = \Delta l_1 + \Delta l_2 \tag{36}$$

Replace the difference Δl_1 or Δl_2 between the hypotenuse and the midline in equation (35) for the solution of included angle, and get, respectively:

$$\sin\alpha = \frac{\left(BP^2 - \Delta l_1^2\right)(\Delta l - \Delta l_1) + \left[BP^2 - (\Delta l - \Delta l_1)^2\right]\Delta l_1}{BP\left[2BP^2 - \Delta l_1^2 - (\Delta l - \Delta l_1)^2\right]} \tag{37}$$

$$\sin\alpha = \frac{\left[BP^2 - (\Delta l - \Delta l_2)^2\right]\Delta l_2 + \left(BP^2 - \Delta l_2^2\right)(\Delta l - \Delta l_2)}{BP\left[2BP^2 - (\Delta l - \Delta l_2)^2 - \Delta l_2^2\right]} \tag{38}$$

And from that we can solve for

$$\Delta l_1 = AB - AP = 0.5\Delta l + \Delta A \tag{39}$$

$$\Delta l_2 = AP - AC = 0.5\Delta l - \Delta A \tag{40}$$

Thereinto, ΔA is

$$\Delta A = 0.5\sqrt{\frac{\Delta l^2(\Delta l + BC\sin\alpha) - 2BC(\Delta l^2\sin\alpha + 0.5BC\cdot\Delta l - 0.5BC^2\sin\alpha)}{\Delta l + BC\sin\alpha}} \tag{27}$$

Obviously, according to the mathematical expression of equations (39) and (40), the two adjacent difference values between hypotenuses and midline can be written in the form of arithmetic series, i.e

$$\Delta l_j = 0.5\Delta l + 2(1.5 - j)\Delta A \qquad (j = 1,2) \tag{41}$$

Thus, from the point of view of mathematical analysis, $2\Delta A$ (or ΔA) is the tolerance of arithmetic series. If based on the path difference of the right side of the equation (39) and the equation (40) , subtract the two equations, can be obtained:

$$2\Delta A = \Delta l_1 - \Delta l_2 \tag{42}$$

Thus, from the point of view of plane geometry, $2\Delta A$ actually represents the difference of two adjacent difference values between hypotenuses and midline.

III.3 **Linear relationship between hypotenuse and midline**

If the two equations are subtracted based on the radial distance in the middle of equations (39) and (40), it can be obtained.

$$\frac{AB + AC}{2} = AP + \Delta A \tag{25}$$

That is, the arithmetic mean of the left and right hypotenuses of a triangle is the sum of the midline and differences of two adjacent differences between the hypotenuses and midline.

IV. Conclusion

This paper deduces a pure mathematical theorem based on the result of engineering theory research. The new quasi-mean value theorem, which reflects the linear relation between the hypotenuse and the midline of a triangle, can be applied to both the mathematical theory and the engineering design. For example, for the engineering field that the author is familiar with, at present, the planar passive positioning must use at least three stations to achieve the detection and positioning of the target. Whether the new quasi-mean value theorem can provide some mathematical support for passive localization of two stations [3].

The existing middle line theorem gives the square relation between three hypotenuses and the middle line. The quasi mean value theorem reveals the linear relationship between the three hypotenuses and the midline in a triangle. Whether this new linear function can help improve the robustness of the physical system. For example, for passive positioning system, whether it can help improve the positioning accuracy of the system. The author has studied how to use the angle median relation among the three stations to improve the ranging accuracy of the detection system based on two-station direction finding.

There are several ways to prove the quasi mean value theorem between the hypotenuse and the midline in a triangle. This article may be just one approach.

Reference

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