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Study of Some Common Fixed Point Theorems in Uniform Spaces

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Abstract. *Our result regarding common fixed point theorems in uniform space generalizes previously established results of Khan[4], Rhoades et al.[6]and Sharma[8] in the sense that our result is obtained for six self-maps selecting a different functional inequality assuming one pair of maps as semi-compatible and another pair is weakly compatible. The result obtained in this paper is substantially different and useful in comparison of previously proved results in the field of uniform spaces.*

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I. Introduction.

Topological spaces are defined as a generalization of metric spaces. In this generalization some features of metric spaces are vanished such as the concept of uniform continuity, uniform convergence and completeness are not defined for arbitrary topological spaces. A generalization of metric spaces was required in which such concepts retrieved. Uniform spaces mount in the middle of metric spaces and general topological spaces. Some useful results were proved by Roy[7], Acharya[1] and Rhoades[5] in uniform spaces. Joshi[2] can be referred for the basic theory, definitions and terminology of uniform spaces.

Referring Khan[4] and Rhoades et al.[6],throughout this paper we assume that (X, **U**) is a sequentially complete Hausdorff uniform space and P is a fixed family of pseudo-metrics on X which generates the uniformity **U** and referring Kelley[3], we assume the following:

(i) $V_{(p, r)} = \{ (x, y) : x, y \in X, p(x, y) < r \}$.

(ii) $G = \left\{ V : V = \bigcap_{i=1}^{n} V_{(p_i, r_i)} : p_i \in P, r_i > 0, i = 1, 2, \ldots, n \right\}$

and for $\alpha > 0$,

(iii) $\alpha V = \left\{ \bigcap_{i=1}^{n} V_{(p_i, \alpha r_i)} : p_i \in P, r_i > 0, i = 1, 2, \ldots, n \right\}$.

Acharya [1] provided the following lemmas 1-4

Lemma 1.[1] If $x, y \in X$, then, for every V in G there is a positive number λ such that $(x, y) \in \lambda V$.

Lemma 2.[1] If $V \in G$ and $\alpha, \beta > 0$, then $\alpha(\beta V) = (\alpha\beta)V$.

Lemma 3.[1] Let p be any pseudo-metric on X and $\alpha, \beta > 0$.

$$\text{If } (x, y) \in \alpha V_{(p, r_1)} o \beta V_{(p, r_2)}, \text{ then } p(x, y) < \alpha r_1 + \beta r_2.$$

Lemma 4.[1] For any arbitrary $V \in G$ there is a pseudo-metric p on X such that $V = V_{(p, 1)}$. This p is called a Minkowski pseudo-metric of V.

The next lemma is essentially due to Khan [4].

Lemma 5. [4] Let $\{y_n\}$ be a sequence in a complete metric space (X, p). If there exists $k \in (0, 1)$ such that $p(y_{n+1}, y_n) \le k\, p(y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.

Rhoades et al. [6] defined compatibility in uniform space which is as follows:

Definition 1. [6] Let A and B be two self-maps of a uniform space, p be a pseudo-metric on X. A and B are said to be compatible on X if

$\lim_{n\to\infty} p(ABx_n, BAx_n) = 0$,whenever $\{x_n\}$ is a sequence in X such that $\{Ax_n\}$ and $\{Bx_n\}$ converge to the same point t in X.

Sharma [8] defined weak compatibility in uniform space which is as follows:

Definition 2.[8] Let A and B be self-mappings of a uniform space, p a pseudo-metric on X. Then the mappings A and B are said to be weakly compatible if they commute at their coincidence point, that is,

$Ax = Bx$ implies $ABx = BAx$ for some $x \in X$.

Now we define semi-compatibility in Uniform spaces

Definition 3. Let P and Q be self-maps of a uniform space, p a pseudo-metric on X. Then a pair of self-maps (P, Q) is said to be semi-compatible if for sequence $\{x_n\}$ in X and $x \in X$, whenever

$\{Px_n\} \to x$, $\{Qx_n\} \to x$, then $PQx_n \to Qx$, as $n\to\infty$, hold.

II. Main Result.

The result proved here in this paper regarding common fixed point theorems in uniform spaces generalizes previously established results of Khan [4], Rhoades et al. [6] and Sharma [8] in the sense that our result is obtained for six self-maps selecting a different functional inequality assuming one pair of maps as semi-compatible and another pair is weakly compatible.

Theorem 2.1. Let P, Q, S, T, A and B be self-mappings of X fulfilling the conditions:

(1) $(Ax, STy) \in V_1$, $(Ax, PQx) \in V_2$ $(By, STy) \in V_3$,$(PQx, STy) \in V_4$, $(PQx, By) \in V_5$ implies that $(Ax, By) \in \alpha_1 V_1 o \alpha_2 V_2 o \alpha_3 V_3 o \alpha_4 V_4 o \alpha_5 V_5$,

Where $\alpha_i = \alpha_i(x, y)$ are non-negative functions from

$X \times X \to [0, 1)$ satisfying $\sup_{x,y\in X} \sum_{i=1}^{5} \alpha_i < 1$ and $\alpha_1 = \alpha_5$;

(2) Either PQ or A is continuous;

(3) $B(X) \subseteq PQ(X)$, $A(X) \subseteq ST(X)$,

(4) Pairs (S,T), (P,Q), (B,T) and (A,Q) are commutative;

(5) (A, PQ) is semi-compatible and (B, ST) is weakly compatible;

Then P, Q, S, T, A and B have a unique common fixed point in X.

Proof. Firstly we assume that $V\in G$ and p be the Minkowski pseudo-metric of V (following by Khan [4]) and Rhoades et al. [6]).

Let $p(Ax, STy) = r_1$, $p(Ax, PQx) = r_2$, $p(By, STy) = r_3$, $p(PQx, STy) = r_4$, $p(PQx, By) = r_5$ for $x, y\in X$.

For any $\varepsilon > 0$, $(Ax, STy) \in (r_1 + \varepsilon)V$, $(Ax, PQx) \in (r_2 + \varepsilon)V$,

$(By, STy) \in (r_3 + \varepsilon)V$, $(PQx, STy) \in (r_4 + \varepsilon)V$, $(PQx, By) \in (r_5 + \varepsilon)V$.

From condition (1) of theorem 2.1 and Lemma 1-3, we get

$p(Ax, By) < \alpha_1(r_1 + \varepsilon) + \alpha_2(r_2 + \varepsilon) + \alpha_3(r_3 + \varepsilon) + \alpha_4(r_4 + \varepsilon) + \alpha_5(r_5 + \varepsilon)$.

Where $\alpha_i = \alpha_i(x, y)$.

Since ε is arbitrary, we have

(6) $p(Ax, By) \le \alpha_1 p(Ax, STy) + \alpha_2 p(Ax, PQx) + \alpha_3 p(By, STy) + \alpha_4 p(PQx, STy) + \alpha_5 p(PQx, By)$.

Now B(X) is contained in PQ(X) and A(X)is contained in ST(X), construct a sequence $\{y_n\}$ in X such that $y_{2n+1} = Bx_{2n+1} = PQx_{2n+2}$ and $y_{2n} = STx_{2n+1} = Ax_{2n}$ for n = 0, 1, 2,….

Firstly we have to prove $\{y_n\}$ is a Cauchy sequence in X.

From (6)

$p(y_{2n}, y_{2n+1}) = p(Ax_{2n}, Bx_{2n+1}) \le \alpha_1 p(Ax_{2n}, STx_{2n+1}) + \alpha_2 p(Ax_{2n}, PQx_{2n}) + \alpha_3 p(Bx_{2n+1}, STx_{2n+1}) + \alpha_4 p(PQx_{2n}, STx_{2n+1}) + \alpha_5 p(PQx_{2n}, Bx_{2n+1})$.

Therefore, $p(y_{2n}, y_{2n+1}) \leq \alpha_2\, p(y_{2n}, y_{2n-1}) + \alpha_3\, p(y_{2n+1}, y_{2n}) + \alpha_4\, p(y_{2n-1}, y_{2n}) + \alpha_5[p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1})]$.

$$p(y_{2n}, y_{2n+1}) \leq \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_3 - \alpha_5}\, p(y_{2n-1}, y_{2n})$$

$$= \lambda\, p(y_{2n-1}, y_{2n}).$$

In general $p(y_n, y_{n+1}) \leq \lambda\, p(y_{n-1}, y_n)$

Now $\lambda < 1$ from (1), so $\{y_n\}$ is a Cauchy sequence in X, hence converges to any z in X, hence we can say that the sub sequences $\{PQx_{2n+2}\}$, $\{STx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ of the Cauchy sequence $\{y_n\}$ also converges to z in X.

Case I. Firstly we assume that A is continuous, implies $APQx_{2n} \to Az$. Since the pair (A, PQ) is semi-compatible, implies $A(PQ)x_{2n} \to PQz$. By uniqueness of limit, we can write Az = PQz.

In (6), substituting $y = x_{2n+1}$ and x = z, we obtain

$p(Az, Bx_{2n+1}) \leq \alpha_1 p(Az, STx_{2n+1}) + \alpha_2\, p(Az, PQz) + \alpha_3 p(Bx_{2n+1}, STx_{2n+1}) + \alpha_4\, p(PQz, STx_{2n+1}) + \alpha_5 p(PQz, Bx_{2n+1})$.

If we take $n \to \infty$ and use above findings, then

$p(Az, z) \leq (\alpha_1 + \alpha_4 + \alpha_5)\, p(Az, z)$.

which implies Az = z. Hence, we conclude that Az = PQz=z.

In (6), substituting $y = x_{2n+1}$ and x = Qz, we obtain

$p(AQz, Bx_{2n+1}) \leq \alpha_1\, p(AQz, STx_{2n+1}) + \alpha_2\, p(AQz, PQQz)$
$+ \alpha_3 p(Bx_{2n+1}, STx_{2n+1}) + \alpha_4 p(PQQz, STx_{2n+1}) + \alpha_5\, p(PQQz, Bx_{2n+1})$.

As (A,Q) and (P,Q) are commutative, therefore PQ(Qz) = Qz and A(Qz) = Qz

Again take $n \to \infty$ and use above findings, then

$p(Qz, z) \leq (\alpha_1 + \alpha_4 + \alpha_5)\, p(Qz, z)$. So that Qz = z. Since PQz = z implies that Pz = z.

Therefore A, P and Q have common fixed point as z.

Since A(X) is contained in ST(X), there exists $v \in X$ such that Az = STv = z

Substituting y = v and $x = x_{2n}$ in (6), assuming $n \to \infty$ and using above findings, we obtain $p(z, Bv) \leq (\alpha_3 + \alpha_5)$ p(Bv, z). Hence z = Bv = STv. Since the pair (B, ST) is weakly compatible. Therefore STBv = BSTv gives that STz = Bz.

Now, substituting y = z and $x = x_{2n}$ in (6), assuming $n \to \infty$ and using above findings, we obtain $p(z, Bz) \leq$ $(\alpha_1 + \alpha_4 + \alpha_5)\, p(z, Bz)$ implies Bz = z. Therefore Bz = z= STz.

Again, substituting y = Tz and $x = x_{2n}$ in (6), we get

$p(Ax_{2n}, BTz) \leq \alpha_1 p(Ax_{2n}, STTz) + \alpha_2 p(Ax_{2n}, PQx_{2n})$
$+ \alpha_3 p(BTz, STTz) + \alpha_4 p(PQx_{2n}, STTz) + \alpha_5\, p(PQx_{2n}, BTz)$.

As (S,T) and (B,T) are commutative. We have ST(Tz) = TS(Tz) = Tz andBTz = Tz.

Assuming $n \to \infty$ and using above findings, we obtain

$$p(z, Tz) \leq (\alpha_1 + \alpha_4 + \alpha_5)\, p(z, Tz).$$

So that Tz = z. Now STz = z gives Sz = z.

Therefore B, S and T have common fixed point as z.

Now Pz = Qz = Sz = Tz = Az = Bz = z shows P,Q,S,T, A and B has a common fixed point z.

Case II. Now we assume that PQ is continuous, gives

$(PQ)^2x_{2n} \to PQz$ and $(PQ)Ax_{2n} \to PQz$. Semi-compatibility of the pair (A, PQ) gives $APQx_{2n} \to PQz$.

Placing $y = x_{2n+1}$ and $x = PQx_{2n}$ in (6), If we take $n \to \infty$ and use previous findings, we obtain $p(PQz, z) \leq$ $(\alpha_1 + \alpha_4 + \alpha_5)\, p(PQz, z)$ implies that PQz = z.

Placing $y = x_{2n+1}$ and x = z in (6),If we take $n \to \infty$ and use previous findings, we obtain $p(Az, z) \leq (\alpha_1 + \alpha_2)\, p(Az, z)$.

so that Az = z. As proved in case I, we can prove here Qz = z. Since PQz = z implies that Pz = z. Thus z is a fixed point of P, Q and A.
Again, as proved in case I, we can prove that z is a fixed point of T, S and B.
Thus z is a common fixed point of P, Q, S, T, A and B.

Now we prove Uniqueness of the fixed point.
Let w be another common fixed point of P, Q, S, T, A and B, then Pw = Qw = Sw = Tw = Aw = Bw = w. Substituting y = w and x = z in (6), we obtain
$p(z, w) \leq (\alpha_1 + \alpha_4 + \alpha_5)\, p(z, w)$, hence z = w. Therefore, z is a common fixed point of P, Q, S, T, A and B which is unique.

Corollary 2.1. Let P, Q, S, T, A and B be self-mappings of X holding the conditions (1), (2), (3), (5) of theorem (2.1) and the pairs (B, ST) and (A, PQ) are compatible.
Then P, Q, S, T, A and B have a unique common fixed point in X.
Proof. Since we know that compatibility indicates weak compatibility, the proof of this corollary follows from theorem 2.1.

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