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Existence and Uniqueness of the Solution of a Mathematical Model of the Transmission Dynamics of Tuberculosis with Exogenous Reinfection, Vaccination, Early Detection and Treatment

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Abstract
Tuberculosis is a disease that has been around for many centuries and despite all the various strategies employed to control the disease, it is still around and has continued to be one of the health challenges to the human races. Various mathematical models have been developed adopting different control strategies. This work combines control strategies of vaccination; early detection and treatment in developing the model. The existence and uniqueness of the solution of the equations were determined using Picard's theorem to establish that the equations of the model are continuous, satisfy Lipchitz condition and are bounded thereby showing that the model equation exist and is unique. Hence it is worthwhile to undertake analyses of the model equations.

Keywords: *Existence, Uniqueness, Solution, Mathematical Model, Exogenous, Reinfection, Vaccination, Early detection, Treatment*

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I. BACKGROUND

Tuberculosis is caused by the bacillus mycobacterium tuberculosis. It has plagued humankind since time immemorial and is presently the second deadliest disease. It is estimated that a third of the world population is infected with one form of the disease or another and about 1.6 million deaths occurred as a result of the disease in 2016. [11].

Although the disease is fatal, only a small proportion of individuals exposed to the disease develop active tuberculosis (approximately 10%) [5], the remaining 90% remain in latency stage for various durations. It affects almost every part of the human organ but the pulmonary type accounts for about 80% [6]. The two ways of progression to the infectious state are endogenous reactivation and exogenous reinfection [3].

Tuberculosis is both preventable and curable. Vaccination with Bacillus Calmatte-Guerine (BCG) at birth is one of the prevention measures but the vaccination wanes with time. The disease can be treated in two ways. The treatment of latent tuberculosis is known as chemoprophylaxis, while the treatment of active tuberculosis is called therapeutics. Generally, treatment of tuberculosis lasts for a long period. [2].

Many researchers have studied the transmission dynamics of tuberculosis with exogenous reinfection using mathematical modeling such as [4], [10], [7], [3] among others. Others such as [8] studied the transmission of the disease with vaccination and screening but without exogenous reinfection. This model is an extension of the work of [4] and [10] by incorporating vaccination and early detection. In addition to the assumptions by [10] and [4], the model further assumes that, as a result of vaccination, some susceptible individuals move to the vaccinated compartment due to the immunity conferred on them. When vaccination wanes, the individuals become susceptible**.** Early detected exposed individuals are treated as such they move to treated compartment and unsuccessfully treated individuals return to the exposed compartment.

II. MODEL DESCRIPTION

The population is partitioned into six compartments; Susceptible $S(t)$, Vaccinated $V(t)$, Exposed $E(t)$, Early detected $E_d(t)$, Infected $I(t)$ and Treated $T(t)$. The susceptible compartment grows as a result of newborns by the population bN, waning of vaccination of the vaccinated class α, and successful treatment of individuals from the treated compartment γ, and loses through contact with the infectious population at the rate βc and natural death rate μ. The vaccinated compartment grows as a result of vaccination of the susceptible population

π and loses α through waning of vaccinated individual and natural death rate μ. Individuals are recruited in to the exposed compartment from the susceptible compartment as a result of contact with infectious individual βc and treatment failure ϕ and loses through exogenous reinfection $\rho\beta c$, early detection of exposed individuals σ, progression to the infected compartment k and natural death μ. The early detected compartment is made up individuals that are moved from the exposed compartment σ, and loses due to movement to the treated compartment q and natural death μ. The infected class gains through the progression of individuals from the exposed class k and exogenous reinfection $\rho\beta c$ and loses through μ, natural death, treatment r and disease induced death d. The treated compartment is made up of those who came in from the early detected class q and from the infected class r while it loses through successful treatment γ, unsuccessful treatment ϕ and natural death μ. Thus the description can be represented by the following diagram and system of differential equations below:

Model Diagram

The model assumptions and description above is represented by the diagram below in figure 1.

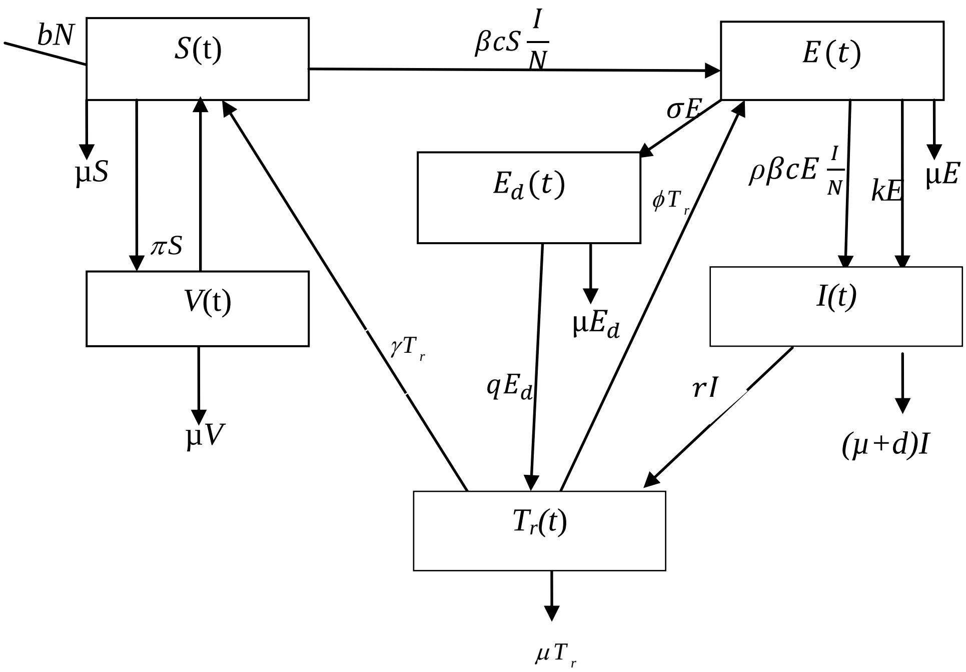

Figure 1: Schematic Representation of the Model

Model Equations

$$\frac{dS}{dt} = bN - \beta cS\frac{I}{N} - (\pi + \mu)S + \alpha V + \gamma T \qquad (1.1)$$

$$\frac{dV}{dt} = \pi S - (\mu + \alpha)V \qquad (1.2)$$

$$\frac{dE}{dt} = \beta cS\frac{I}{N} + \phi T - \rho\beta cE\frac{I}{N} - (\mu + k + \sigma)E \qquad (1.3)$$

$$\frac{dE_d}{dt} = \sigma E - (q + \mu)E_d \qquad (1.4)$$

$$\frac{dI}{dt} = \rho\beta cE\frac{I}{N} + kE - (\mu + r + d)I \qquad (1.5)$$

$$\frac{dT}{dt} = rI + qE_d - (\mu + \phi + \gamma)T \qquad (1.6)$$

$$N = S + V + E + E_d + I + T \tag{1.7}$$

$S(0) > 0, V(0) \geq 0, E(0) \geq 0\ E_d(0) \geq 0, I(0) \geq 0, T(0) \geq 0$

With the initial conditions;

$$N(0) = N_0, S(0) = S_0, V(0) = V_0, E(0) = E_0, E_d(0) = E_{d0}, I(0) = I_0, T(0) = T_0 \tag{1.8}$$

Dimensionless Transformations

$$\frac{dN}{dt} = bN - \mu N - dI \tag{1.9}$$

Let, $s = \frac{S}{N}$, $v = \frac{V}{N}$, $e = \frac{E}{N}$, $e_d = \frac{E_d}{N}$, $i = \frac{I}{N}$ and $t = \frac{T}{N}$

Then we have

$$s + v + e + e_d + i + t = 1 \tag{1.10}$$

Differentiating (1.9) gives

$$N' = bN - \mu N - dI$$

$$\frac{N'}{N} = b - \mu - di \tag{1.11}$$

Differentiating $s = \frac{S}{N}$ yields

$$s' = \frac{S'}{N} - s\frac{N'}{N} \tag{1.12}$$

Substituting (1.1) and (1.10) into (1.11) gives

$$s' = b - \beta csi - (\pi + b)s + \alpha v + \gamma t + dis \tag{1.13}$$

Applying the same procedure transforms equations (1.1)-(1.6) as given by equation (1.14) below

$$\begin{aligned}
s' &= b - \beta csi - (b + \pi)s + \alpha v + \gamma t + dis \\
v' &= \pi s - (\alpha + b)v + div \\
e' &= \beta csi + \phi t - \rho\beta cei - (b + k + \sigma)e + die \\
e_d{}' &= \sigma e - (b + q)e_d + die_d \\
i' &= \rho\beta cei + ke - (b + r + d)i + di^2 \\
t' &= ri + qe_d - (b + \gamma + \phi) + dit
\end{aligned} \tag{1.14}$$

$$s + v + e + e_d + i + t = 1 \tag{1.15}$$

$$e = 1 - s - v - e_d - i - t_r \tag{1.16}$$

Substituting (1.16) into (1.14) yields

$$s' = b - \beta csi - (b + \pi)s + \alpha v + \gamma t + dis \tag{1.17}$$

$$v' = \pi s - (\alpha + b)v + div \tag{1.18}$$

$$e_d' = \sigma(1 - s - v - e_d - i - t_r) - (b + q)e_d + die_d \tag{1.19}$$

$$i' = \rho\beta ci(1 - s - v - e_d - i - t_r) + k(1 - s - v - e_d - i - t_r) - (b + r + d)i + di^2 \tag{1.20}$$

$$t_r' = ri + qe_d + \alpha v - (b + \gamma + \phi)t_r + dit_r \tag{1.21}$$

$$s + v + e + e_d + i + t_r = 1 \tag{1.22}$$

$$t_r = 1 - s - v - e_d - i \tag{1.23}$$

Substituting (1.23) into (1.17)-(1.21) gives

$$s' = b - \beta csi - (b + \pi)s + \alpha v + \gamma(1 - s - v - e_d - i) + dis \tag{1.24}$$

$v' = \pi s - (\alpha + b)v + div$ (1.25)

$e_d{}' = -(b + q)e_d + die_d$ (1.26)

$i' = -(b + r + d)i + di^2$ (1.27)

III. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THE MODEL EQUATIONS

In this section we establish the conditions for the existence and uniqueness of the transformed model equations (1.24-1.27) using the following method in [9].

Theorem 2.1: Picard's Theorem

Suppose $f(t,x)$ is continuous and satisfies a Lipschitz condition in the closed and bounded domain $\|x - x_0\| \leq \varphi, \|t - t_0\| \leq \tau$, then any equation of the form

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0 \qquad (2.1)$$

called an initial value problem has a unique solution in the interval $\|t - t_0\| \leq h$

where $h = \left\{\tau, \frac{\varphi}{M}\right\}$. . [1]

The model equations will be transformed into the form (2.1) and the theorem above will be used to prove the existence and uniqueness of the transformed equations.

Consider equations (1.24)-(1.27),

Let

$x = [s, v, e, e_d, i, t_r]^T$

$$f(t,x) = [f_1(t.x), f_2(t,x), f_3(t,x), f_4(t,x)]^T \qquad (2.2)$$

Where

$$f_1(t,x) = b - \beta csi - (b + \pi)s + \alpha v + \gamma(1 - s - v - e_d - i) + dis$$

$$f_2(t,x) = \pi s - (\alpha + b)v + div$$

$$f_3(t,x) = -(b + q)e_d + die_d \qquad (2.3)$$

$$f_4(t,x) = -(b + r + d)i + di^2$$

The model equations are now of the form (2.1).

Suppose the function $f(t,x)$ is defined and continuous in x and satisfies a Lipchitz condition in the closed and bounded region

Let

$$D = \{x = (s, v, e_d, i,) : s, v, e_d, i, \leq 1\} \qquad (2.4)$$

Define

$\|x - x_0\| \leq \varphi, \|t\| \leq \tau, t_0 = 0, x_0 = (s_0, v_0, e_{d0}, i_0,)$

We shall prove using Picard's theorem that the solution of (1.24)-(1.27) exists and is unique, by proving the following:

1. f is continuous
2. f satisfies Lipchitz condition
3. $|f| \leq M$

$f(t,x)$ is continuous since each component of $f_i = 1,2,3,4$ is a continuous function of the variable $x = [s, v, e, e_d, i, t_r]^T$.

To establish the Lipchitz condition for the model equations,

Let $y = [s_*, v_*, e_{d*}, i_*]^T$

$f(y) = [f_1(y), f_2(y), f_3(y), f_4(y)]^T$

Noting that $s, v, e, e_d, i, t_r \le 1$

We have

$$\left| f_1(x) - f_1(y) \right| = \left| b - \beta csi - (b+\pi)s + \alpha v + \gamma(1 - s - v - e_d - i) + dis \right|$$

$$= \left| b - \beta c[i(s - s_*) + s(i - i_*)] + [-(b+\pi)(s - s_*)] + \alpha(v - v_*) + \gamma + \right.$$

$$\left. [-\gamma(s - s_*)] + [-\gamma(v - v_*)] + [-\gamma(e_d - e_{d*})] + [-\gamma(i - i_*)] \right|$$

$$\le \left| b + \gamma + \beta c|i||s - s_*| + \beta c|s||i - i_*| + b|s - s_*| + \pi|s - s_*| + \alpha|v - v_*| \right.$$

$$\left. \gamma|s - s_*| + \gamma|v - v_*| + \gamma|e_d - e_{d*}| + \gamma|i - i_*| + d|s - s_*| + d|i - i_*| \right|$$

$$= b + \gamma + (b + \pi + \gamma + \beta c)|s - s_*| + (\alpha + \gamma)|v - v_*| + \gamma|e_d - e_{d*}| +$$

$$(\beta c + d + \gamma)|i - i_*|$$

$$= l_{11}|s - s_*| + l_{12}|v - v_*| + l_{13}|e_d - e_{d*}| + l_{14}|i - i_*|$$

$$\left| f_1(x) - f_1(y) \right| \le L_1 \|x - y\|$$

Where,

$L_1 = \max\{l_{11}, l_{12}, l_{13}, l_{14}\}$ and $l_{11} = b + \beta c + \pi + \gamma$, $l_{12} = \alpha + \gamma$, $l_{13} = \gamma$, $l_{14} = \beta c + d + \gamma$ are constants depending on the parameters of the model.

$$\left| f_2(x) - f_2(y) \right| = \left| \pi(s - s_*) + (-\alpha - b)(v - v_*) + d[i(v - v_*) + v(i - i_*)] \right|$$

$$\le \pi|s - s_*| + (\alpha + b)|v - v_*| + d|v - v_*| + d|i - i_*|$$

$$= \pi|s - s_*| + (\alpha + b + d)|v - v_*| + 0|e_d - e_{d*}| + d|i - i_*|$$

$$= l_{21}|s - s_*| + l_{22}|v - v_*| + l_{23}|e_d - e_{d*}| + l_{24}|i - i_*|$$

$$\left| f_2(x) - f_2(y) \right| \le L_2 \|x - y\|$$

Where,

$L_2 = \max\{l_{21}, l_{22}, l_{23}, l_{24}\}, l_{21} = \pi, l_{22} = \alpha + b + d, l_{23} = 0, l_{24} = 0$ are constants depending on the parameters of the model.

$$\left| f_3(x) - f_3(y) \right| = \left| (-b - q)(e_d - e_{d*}) + d[i(e_d - e_{d*}) + e_d(i - i_*)] \right|$$

$$\le (b + q)|e_d - e_{d*}| + d|i||e_d - e_{d*}| + d|e_d||i - i_*|$$

$$= (b + d + q)|e_d - e_{d*}| + d|i - i_*|$$

$$= 0|s - s_*| + 0|v - v_*| + (b + d + q)|e_d - e_{d*}| + d|i - i_*|$$

$$= l_{31}|s - s_*| + l_{32}|v - v_*| + l_{33}|e_d - e_{d*}| + l_{34}|i - i_*|$$

$$\left| f_3(x) - f_3(y) \right| \le L_3 \|x - y\|$$

Where,

$L_3 = \max\{l_{31}, l_{32}, l_{33}, l_{34}\}, l_{31} = 0, l_{32} = 0, l_{33} = b + d + q, l_{34} = d$ are constants depending on the parameters of the model.

$$\left| f_4(x) - f_4(y) \right| = \left| (-b - r -)d(i - i_*) + d[i(i - i_*) + i(i - i_*)] \right|$$

$$\le (b + r + d)|i - i_*| + d|i - i_*|$$

$$= (b + r + d)|i| + d|i|$$

$$= (2b + r + d)|i|$$

$$= 0|s - s_*| + 0|v - v_*| + 0|e_d - e_{d*}| + (2b + r + d)|i - i_*|$$

$$= l_{41}|s - s_*| + l_{42}|v - v_*| + l_{43}|e_d - e_{d*}| + l_{44}|i - i_*|$$

$$\left|f_4(x)-f_4(y)\right| \le L_4\left\|x-y\right\|$$

Where,

$L_4=\max\{l_{41},l_{42},l_{43},l_{44}\}, l_{41}=0, l_{42}=0, l_{43}=0, l_{44}=2b+r+d$ are constants depending on the parameters of the model.

Therefore,

$$\begin{aligned}\left\|f(x)-f(y)\right\| &= \max_{1\le i\le 4}\left\{\left|f_1(x)-f_1(y)\right|+\ldots+\left|f_6(x)-f_6(y)\right|\right\}\\ &= \max_{1\le i\le 4} L_i\left\|x-y\right\|, i=1,2,3,4\\ &\le L\left\|x-y\right\|\end{aligned}$$

Where,

$$L=\max\{L_1,L_2,L_3,L_4\}$$

To obtain the bound for *f*, noting that, $s,v,e_d,i,\le 1$ we have,

$$\begin{aligned}\left|f_1(x)\right| &\le b+\beta c|s||i|+(\pi+b)|s|+\alpha|v|+\gamma+\gamma|s|+\gamma|v|+\gamma|e_d|+\gamma|i|+d|i||s|\\ &= 2b+\beta c+\alpha+\pi+5\gamma+d\\ &= M_1\end{aligned}$$

Similarly,

$$\begin{aligned}\left|f_2(x)\right| &\le \pi+\alpha+b+d\\ &= M_2\end{aligned}$$

$$\begin{aligned}\left|f_3(x)\right| &\le b+q+d\\ &= M_3\end{aligned}$$

$$\begin{aligned}\left|f_4(x)\right| &\le b+r+2d\\ &= M_4\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|f(x)\right\| &= \max\{M_1,M_2,M_3,M_4,M_5,M_6,\}\\ &\le M\end{aligned}$$

Thus, there exists a unique solution for the IVP in some interval $h=\min\left\{\tau,\dfrac{\varphi}{M}\right\}$.

IV. DISCUSSION AND CONCLUSION

In this paper, a mathematical model for the transmission dynamics of tuberculosis with exogenous reinfection was formulated using the approach by [4] and [10] incorporating vaccination and early detection and ascertained the existence and uniqueness of its solution. It is shown that the system of equations represent a useful mathematical model of a physical system by carrying out a classical qualitative proof of the existence and uniqueness of the solution to the governing system of model equations. This therefore has shown that it is worthwhile to undertake the analyses of the model equations since the solution exists and is unique.

References

[1]. Ademola, T. A. and Ogundiran, M. O. (2007). On the Existence and Uniqueness of Solutions of a Generalized Lipschitz Ordinary Differential Equations. Ife Journal of Science vol. 9, no. 2 (2007).

[2]. Anggriani, N., Fithriati, I. N., Agustian, D. and Supriatna, A. K. (2015). The Effects of DOTS on the Transmission of Tuberculosis with Endogenous Reactivation and Exogenous Reinfection Factors. 3rd International Conference on Chemical, Agricultural and Medical sciences (CAMS-2015) Dec. 10-11, 2015, Singapore.

[3]. Borgdorff, M. W., Geskus, R. B., Kremer, K., Kalisvaart, N. and van Solingen, D. (2011). The Incubation Period Distribution of Tuberculosis Estimated with a Molecular Epidemiological Approach. *Int. J Epidemiol;* 40(4):964-970

[4]. Dang, B.C., Musa, S., Adamu, A., Tumba, M.V. and Wangercha, W.A. (2013). On Models of Tuberculosis with Exogenous Re-infection. *Journal of Natural Sciences* 3(12).

[5]. Ewa, A. U, Essiet, D. F. and Monu, S. J. U. (2015). Tuberculosis in Children Living amongst Adults with Tuberculosis and Leprosy Referral Hospital, Eku, Nigeria. *Journal of Tuberculosis Research,* **3**, 80-89.

[6]. Kalu, A. U. and Inyama, S. C. (2012). Mathematical model of the Role of Vaccination and Treatment on the Transmission Dynamics of Tuberculosis. *Gen. Math. Notes, Vol. 11, No. 1, July 2012, pp. 10-23 ISSN 2219-7184*; Copyright © ICSRS Publication, 2012 www.i-csrs.org http://dx.doi.org10.106/j.phrp2014 .01.002

[7]. National Tuberculosis and Leprosy Control Programme (2010). Federal Ministry of Health, Nigeria, Department of Public Health. Workers manual- Revised 5th edition.

[8]. Nyerere, N., Luboobi, L. S.and Nkansah-Gyekye, Y. (2014). Modeling the Effect of Screening and Treatment on the Transmission of Tuberculosis Infections. *Mathematical Theory and Modeling* ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online) 4, (7), www.iiste.org.

[9]. Udoo, I. J. M., Kimbir R. A. and Odekunle, R. M. (2015). Existence and Uniqueness of Solution of an HIV/AIDS Model Considering Counseling, Vaccination and Antiretroviral Therapy (ART). Mathematical Theory and Modeling www.iiste.org ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online) Vol.5, No.11, 2015.

[10]. Wangercha, W.A., Kimbir, A. R., Musa S. and Adamu, A. (2013). Modeling the Effect of Variation Rate on the Transmission Dynamics of Tuberculosis. *IOSR Journal of Mathematics, 6(3),* 01-09.

[11]. World Health Organization (2013). Systematic Screening for Active Tuberculosis -Principles and Recommendations Geneva. www.who.int/tb/tbscreening