International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321- 4767 P-ISSN: 2321-4759
www.ijmsi.org Volume 8 Issue 9 || September, 2020 || PP 46-55

Periodic solutions for a second order nonlinear functional differential equations with impulses

Lili Wang

School of Mathematics and Statistics, Anyang Normal University, Anyang Henan 455000, China

Abstract:
The second order impulsive functional differential equations with periodic coefficients

$$\begin{cases} x''(t)+a(t)x'(t)+b(t)x(t)=\lambda c(t)f(t,x(t),x(t-\tau(t))), & t\neq t_j, \\ \Delta x\big|_{t=t_j}=I(x(t_j)), \quad -\Delta x'\big|_{t=t_j}=J_j(x(t_j)), & t=t_j,\ j\in z^{+}. \end{cases}$$

is considered in this work. By using Krasnoselskii's fixed point theorem, we establish some criteria for the existence of periodic solutions to the delay impulsive differential equations.

Keywords: *Periodic solution; Delay differential equations; Fixed point theorem; Impulse.*

Date of Submission: 15-12-2020 Date of Acceptance: 30-12-2020

I. INTRODUCTION

In recent years, impulsive and periodic boundary value problems have been studied extensively in the literature, see [1-9]. In [2,4,5,10], periodic boundary value problems were studied extensively. Jiang [4] has applied Krasnoselskii's fixed point theorem to establish the existence of positive solution to problem

$$\begin{cases} -x''+Mx=f(t,x), & t\in[0,2\pi] \\ x(0)=x(2\pi),\ x'(0)=x'(2\pi) \end{cases} \tag{1.1}$$

he proved that there exists at least one positive solution. Zhang and Wang [10] studied (1.1) for singularity. They gave the existence of multiple positive solutions via the Krasnoselskii's fixed point theorem.

On the other hand, impulsive differential equations were studied extensively. In [6,8,9], authors used the method of lower and upper solutions with monotone iterative technique to study impulsive differential equations. In [1,7], authors used the Krasnoselskii's fixed point theorem in a cone to impulsive differential equations and obtained the existence of positive solutions.

Motivated by the above works, in this paper, we shall deal with the existence of a class of higher-dimensional of second order impulsive functional differential equations with periodic coefficients

$$\begin{cases} x''(t)+a(t)x'(t)+b(t)x(t)=\lambda c(t)f(t,x(t),x(t-\tau(t))), & t\neq t_j, \\ \Delta x\big|_{t=t_j}=I(x(t_j)),\ -\Delta x'\big|_{t=t_j}=J_j(x(t_j)), & t=t_j,\ j\in z^{+}, \end{cases} \tag{1.2}$$

Here,

(A1) $a,b:R\to R^{+}$, $c,\tau:R\to R$ are all continuous T-periodic functions, and $\int_0^T a(s)ds>0$, $\int_0^T b(s)ds>0$, $\tau'(t)\neq 1$, for all $t\in[0,T]$;

(A2) $f : R^3 \to R$ is continuous for any $(t, x, y) \in R^3$ and is T -periodic in t for all $(x, y) \in R^2$.

(A3) There exist positive constants L and E such that

$$\left| f(t, x, y) - f(t, z, w) \right| \le L \left| x - z \right| + E \left| y - w \right|.$$

(A4) $I_k \in C(R^+, R)$, $J_k \in C(R^+, R^+)$ with a constant m such that $-\frac{1}{m} J_k(x) < I_k(x) < \frac{1}{m} J_k(x)$,

and $\Delta x\big|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $-\Delta x'\big|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$, where $x(t_j^+)$ and $x(t_j^-)$ represent the right and the left limit of $x(t_j)$, there exist an integer $p > 0$ such that $t_{j+p} = t_j + T$, $I_{j+p} = I_j$, $j \in z^+$.

For convenience, we first introduce the related definition and the fixed point theorem applied in the paper.

Definition 1.1 Let X be a Banach space and K be a closed nonempty sunset of X, K is a cone if

(1) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \ge 0$;

(2) $u, -u \in K$ imply $u = 0$.

Theorem 1.1 (Krasnoselskii [11]) Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are open bounded subsets of X with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let

$$\phi : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(1) $\|\phi y\| \le \|y\|, \forall y \in K \cap \partial\Omega_1$ and $\|\phi y\| \ge \|y\|, \forall y \in K \cap \partial\Omega_2$; or

(2) $\|\phi y\| \ge \|y\|, \forall y \in K \cap \partial\Omega_1$ and $\|\phi y\| \le \|y\|, \forall y \in K \cap \partial\Omega_2$.

Then ϕ has a fixed point in $K \cap (\overline{\Omega_2} \setminus K \cap \partial\Omega_1)$.

In this paper we always assume that

(H1) $f(t, \xi, \eta) \ge 0$ for all $(t, \xi, \eta) \in R \times BC(R, R_+) \times R_+$.

II. PRELIMINARIES

In order to define the solution of (1.2) we consider the following Banach spaces:

$$PC(R, R) = \{x : R \to R : x\big|_{(t_j, t_{j+1})} \in C(t_j, t_{j+1}), x(t_j^-) = x(t_j), \exists x(t_j^+), j \in z^+\}$$

is a Banach space with the norm $\|x\|_{PC} = \sup_{t \in [0,T]} \sum_{j=1}^{n} \left| x_j(t) \right|$.

$$PC^1(R, R) = \{x : R \to R : x\big|_{(t_k, t_{k+1})}, x'\big|_{(t_k, t_{k+1})} \in C(t_k, t_{k+1}), x(t_k^-) = x(t_k), x'(t_k^-) = x'(t_k), \exists x(t_k^+), x(t_k), j \in z^+\}$$

is also a Banach space with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

Lemma 2.1. ([12]) Suppose that (A1, A4) holds and

$$\frac{R_1[\exp(\int_0^T a(u)du)-1]}{Q_1 T} \ge 1, \tag{2.1}$$

$$R_1 = \max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp(\int_t^s a(u)du)}{\exp(\int_0^T a(u)du)-1}b(s)ds\right|, Q_1 = \left(1+\exp(\int_0^T a(u)du)\right)^2 R_1^2,$$

there exist continuous T -periodic functions p and q such that $q(t) > 0$, $\int_0^T p(u)du > 0$, and

$$p(t)+q(t) = a(t), q'(t) + p(t)q(t) = b(t) \quad \text{for all } t \in R.$$

Therefore

$$p(t)+q(t) = a(t), q'(t) + p(t)q(t) = b(t), t \in R.$$

Lemma 2.2. ([13]) Suppose the conditions of Lemma 2.1 hold and $\varphi(t) \in X$. The equation

$$x''(t) + a(t)x'(t) + b(t)x(t) = \varphi(t) \tag{2.2}$$

has a T -periodic solution. Moreover, the periodic solutions can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)\varphi(s)ds, \tag{2.3}$$

where

$$G(t,s) = \frac{\int_t^s \exp[\int_t^u q(v)dv + \int_u^s p(v)dv]du + \int_s^{t+T}\exp[\int_t^u q(v)dv + \int_u^{s+T} p(v)dv]du}{[\exp(\int_0^T p(u)du)-1][\exp(\int_0^T q(u)du)-1]}.$$

Therefore, the equation $x''(t) + a(t)x'(t) + b(t)x(t) = \lambda c(t) f(t, x(t), x(t-\tau(t)))$ has a T -periodic solution, it can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)\lambda c(s) f(s, x(s), x(s-\tau(s)))ds,$$

and by (H1), we have

$$G(t,s)\lambda c(s) f(s, x(s), x(s-\tau(s))) \ge 0, (t,s) \in R^2.$$

The following lemma is fundamental to our discussion. Since the method is similar to that in the literature [14], we omit the proof.

Lemma 2.3. $x \in X$ is a solution of (1.2) if and only if $x \in X$ is a solution of the equation

$$x(t) = \int_t^{t+T} G(t,s)\lambda C(s) f(s, x(s), x(s-\tau(s)))ds + \sum_{j:t_j \in [t,t+T]} G(t,t_j)J_j(x(t_j))$$
$$+ \sum_{j:t_j \in [t,t+T]} \frac{\partial G(t,s)}{\partial s}\bigg|_{s=t_j} I_j(x(t_j)). \tag{2.4}$$

Corollary 2.1. Green's function $G(t,s)$ satisfies the following properties:

$$G(t, t+T) = G(t,t), \quad G(t+T, s+T) = G(t,s),$$

$$\frac{\partial}{\partial s} G(t,s) = p(s)G(t,s) - \frac{\exp \int_t^s q(v)dv}{\exp \int_0^T q(v)dv - 1},$$

$$\frac{\partial}{\partial t} G(t,s) = -q(s)G(t,s) + \frac{\exp \int_t^s p(v)dv}{\exp \int_0^T p(v)dv - 1}.$$

Lemma 2.4. ([13]) Let $A = \int_0^T a(u)du$, $B = T^2 \exp(\frac{1}{T}\int_0^T \ln b(u)du)$. If $A^2 \geq 4B$, (2.5)

then

$$\min\left\{\int_0^T p(u)du, \int_0^T q(u)du\right\} \geq \frac{1}{2}(A - \sqrt{A^2 - 4B}) := l,$$
$$\max\left\{\int_0^T p(u)du, \int_0^T q(u)du\right\} \leq \frac{1}{2}(A + \sqrt{A^2 - 4B}) := m.$$

Therefore the function $G(t,s)$ satisfies

$$0 < N_1 =: \frac{T}{(e^m - 1)^2} \leq G(t,s) \leq \frac{T\exp(\int_0^T a(u)du)}{(e^l - 1)^2} := M_1, s \in [t, t+T],$$
$$1 \geq \frac{G(t,s)}{M_1} \geq \frac{N_1}{M_1} = \sigma.$$

Now, before presenting our main results, we give the following assumptions.

(H2) $f(t, \phi(t), \phi(t - \tau(t)))$ is a continuous function of t for each $\phi \in BC(R, R^+)$.

(H3) For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\{\phi, \psi \in BC, \|\phi\| \leq L, \|\psi\| \leq L, \|\phi - \psi\| < \delta, 0 \leq s \leq T\}$$

imply $\left| f(s, \phi(s), \phi(s - \tau(s))) - f(s, \psi(s), \psi(s - \tau(s)))\right| < \varepsilon$.

III. MAIN RESULTS

For every positive solution of (1.2), one has

$$\|x\| = \sup_{t\in[0,T]}\{|x(t)|, x \in X\}.$$

Let K is a cone in X, which is defined as

$$K = \left\{x \in X : x(t) \geq \sigma \|x\|, t \in [0,T]\right\}.$$

Now we define a mapping $T : K \to K$,

$$(Tx)(t) = \int_t^{t+T} G(t,s)\lambda C(s) f(s,x(s),x(s-\tau(s)))ds + \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j))$$

$$+ \sum_{j:t_j\in[t,t+T]} \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_j} I_j(x(t_j)),$$

then we have

$$(Tx)(t) = \int_t^{t+T} G(t,s)\lambda C(s) f(s,x(s),x(s-\tau(s)))ds + \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j))$$

$$+ \sum_{j:t_j\in[t,t+T]} \left(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}\right) I_j(x(t_j))$$

$$= \int_t^{t+T} G(t,s)\lambda C(s) f(s,x(s),x(s-\tau(s)))ds + \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j))$$

$$+ \sum_{j:t_j\in[t,t+T]} G(t,t_j)p(t_j)I_j(x(t_j)) - \sum_{j:t_j\in[t,t+T]} \left(\frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}\right) I_j(x(t_j)).$$

Lemma 3.1. $T : K \to K$ is well-defined.

Proof. For each $x \in K$, by (H2) we have $(Tx)(t)$ is continuous and

$$(Tx)(t+T) = \int_{t+T}^{t+2T} G(t,s)\lambda C(s) f(s,x(s),x(s-\tau(s)))ds + \sum_{j:t_j\in[t,t+T]} G(t+T,t_j+T)J_j(x(t_j+T))$$

$$+ \sum_{j:t_j\in[t,t+T]} \left(p(t_j+T)G(t+T,t_j+T) - \frac{\exp\int_{t+T}^{t_j+T} q(v)dv}{\exp\int_0^T q(v)dv - 1}\right) I_j(x(t_j+T))$$

$$= \int_t^{t+T} G(t+T,v+T)\lambda C(v+T) f(v+T,x(v+T),x(v+T-\tau(v+T)))dv$$

$$+ \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j)) + \sum_{j:t_j\in[t,t+T]} \left(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}\right) I_j(x(t_j))$$

$$= \int_t^{t+T} G(t,v)\lambda C(v) f(v,x(v),x(v-\tau(v)))dv + \sum_{j:t_j\in[t,t+T]} G(t,t_j)I_j(x(t_j))$$

$$+ \sum_{j:t_j\in[t,t+T]} \left(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}\right) I_j(x(t_j))$$

$$= (Tx)(t).$$

Thus, $Tx \in PC(J,R)$, since

$$N_1 \le G(t,s) \le M_1, \quad s \in [t, t+T],$$

and
$$\left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_j} = p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}, \quad t_j \in [t,t+T],$$

$$N_2 \le \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_j} \le M_2, \quad t_j \in [t,t+T].$$

We define $M = \max\{M_1, M_2\}, N = \min\{N_1, N_2\}$.

Hence, for $x \in K$, we have

$$\|Tx\| \le M\left(\int_0^T \left|\lambda c(s) f(s, x(s), x(s-\tau(s)))\right| ds + \sum_{j:t_j \in [t,t+T]} J_j(x(t_j)) + \sum_{j:t_j \in [t,t+T]} I_j(x(t_j))\right), \qquad (3.1)$$

and

$$\begin{aligned}(Tx)(t) &\ge N\left(\int_0^T \left|\lambda c(s) f(s, x(s), x(s-\tau(s)))\right| ds + \sum_{j:t_j \in [t,t+T]} J_j(x(t_j)) + \sum_{j:t_j \in [t,t+T]} I_j(x(t_j))\right) \\ &= \frac{N}{M} M\left(\int_0^T \left|\lambda c(s) f(s, x(s), x(s-\tau(s)))\right| ds + \sum_{j:t_j \in [t,t+T]} J_j(x(t_j)) + \sum_{j:t_j \in [t,t+T]} I_j(x(t_j))\right) \\ &\ge \sigma \|Tx\|.\end{aligned}$$

Therefore, $Tx \in K$. This completes the proof.

Lemma 3.2. $T: K \to K$ is completely continuous.

Proof. We first show that T is continuous .

By (H3), for any $L > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$\{\phi, \psi \in BC, \|\phi\| \le L, \|\psi\| \le L, \|\phi - \psi\| \le \delta\}$ imply

$$\sup_{0 \le s \le T} \left| f(s, \phi(s), \phi(s-\tau(s))) - f(s, \psi(s), \psi(s-\tau(s)))\right| < \frac{\varepsilon}{2\lambda MTC},$$

where $C = \max_{0 \le t \le T} |c(t)|$.

Since $J_j, I_j \in C(R,R)$, we have $\left|J_j(\phi) - J_j(\psi)\right| < \frac{\varepsilon}{4Mp}$, $\left|I_j(\phi) - I_j(\psi)\right| < \frac{\varepsilon}{4Mp}$.

If $x, y \in K$ with $\|x\| \le L, \|y\| \le L, \|x - y\| \le \delta$, then

$$\begin{aligned}\left|(Tx)(t)-(Ty)(t)\right| &\le \int_t^{t+T}\left|G(t,s)\right|\left|\lambda c(s)f(s,x(s),x(s-\tau(s)))-\lambda c(s)f(s,y(s),y(s-\tau(s))\right|ds\\ &\quad+\sum_{j:t_j\in[t,t+T]}\left|G(t,t_j)\right|\left|J_j(x(t_j))-J_j(y(t_j))\right|+\sum_{j:t_j\in[t,t+T]}\left|\frac{\partial G(t,s)}{\partial s}\right|_{s=t_j}\left|I_j(x(t_j))-I_j(y(t_j))\right|\\ &\le \lambda MC\int_0^T\left|G(t,s)\right|\left|f(s,x(s),x(s-\tau(s))-f(s,y(s),y(s-\tau(s))\right|ds\\ &\quad+M\sum_{j=1}^{P}\left|J_j(x(t_j))-J_j(y(t_j))\right|+M\sum_{j=1}^{P}\left|I_j(x(t_j))-I_j(y(t_j))\right|\\ &< M\lambda TC\frac{\varepsilon}{2M\lambda TC}+2Mp\frac{\varepsilon}{4Mp}=\varepsilon\end{aligned}$$

for all $t\in[0,T]$, this yields $\left\|Tx-Ty\right\|<\varepsilon$, thus T is continuous.

Next we show that T maps any bounded sets in K into relatively compact sets. Now we first prove that f maps bounded sets into bounded sets. Indeed, let $\varepsilon=1$, by (H3), for any $\mu>0$, there exists $\delta>0$ such that $\{x,y\in BC,\left\|x\right\|\le\mu,\left\|y\right\|\le\mu,\left\|x-y\right\|\le\delta,0\le s\le T\}$ imply

$$\left|f(s,x(s),x(s-\tau(s))-f(s,y(s),y(s-\tau(s))\right|<1.$$

Choose a positive integer N such that $\frac{\mu}{N}<\delta$. Let $x\in BC$ and define

$$x^k(t)=\frac{x(t)k}{N},k=0,1,2\cdots,N.$$

If $\left\|x\right\|<\mu$, then

$$\left\|x^k-x^{k-1}\right\|=\sup_{t\in R}\left|\frac{x(t)k}{N}-\frac{x(t)(k-1)}{N}\right|\le\left\|x\right\|\frac{1}{N}\le\frac{\mu}{N}<\delta.$$

Thus,

$$\left|f(s,x^k(s),x^k(s-\tau(s))-f(s,x^{k-1}(s),x^{k-1}(s-\tau(s))\right|<1$$

for all $s\in[0,T]$, this yields

$$\begin{aligned}&\left|f(s,x(s),x(s-\tau(s))\right|_0=\left|f(s,x^N(s),x^N(s-\tau(s))\right|\\ &\le\sum_{k=1}^{N}\left|f(s,x^k(s),x^k(s-\tau(s))-f(s,x^{k-1}(s),x^{k-1}(s-\tau(s))\right|+\left|f(s,0,0)\right| \qquad (3.2)\\ &<N+\left\|f\right\|=:W,\end{aligned}$$

and

$$\left|J_j(x(t_j))\right| = \left|J_j(x^N(t_j))\right| \le \sum_{k=1}^{N}\left|J_j(x^N(t_j)) - J_j(x^{N-1}(t_j))\right| + \left|J_j(0)\right| \le N + \left|J_j(0)\right| := U_1,$$

$$\left|I_j(x(t_j))\right| = \left|I_j(x^N(t_j))\right| \le \sum_{k=1}^{N}\left|I_j(x^N(t_j)) - I_j(x^{N-1}(t_j))\right| + \left|I_j(0)\right| \le N + \left|I_j(0)\right| := U_2,$$

we define $U = \max\{U_1, U_2\}$.

It follows from (3.1) that for $t \in [0, T]$,

$$\begin{aligned}\|Tx\| &= \sup_{t\in R}\left|(Tx)(t)\right| \\ &\le M\lambda C\int_0^T\left|f(s, x(s), x(s-\tau(s)))\right|ds + M\left(\sum_{j:t_j\in[t,t+T]}\left|I_j(x(t_j))\right| + \sum_{j:t_j\in[t,t+T]}\left|J_j(x(t_j))\right|\right) \\ &\le M\lambda CTW + 2MpU.\end{aligned}$$

Finally, for $t \in R$, we have

$$\begin{aligned}(Tx)'(t) &= \int_t^{t+T}\left[-q(s)G(t,s) + \frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv - 1}\right]\lambda c(s)f(s, x(s), x(s-\tau(s)))ds \\ &+ \sum_{j=1}^{p}\left(-q(s)G(t,s) + \frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv - 1}\right)J_j(x(t_j)) \\ &+ \sum_{j=1}^{p}\left(p(t_j)\left(-q(t_j)G(t,t_j) + \frac{\exp\int_t^{t_j} p(v)dv}{\exp\int_0^T p(v)dv - 1}\right) + \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}q(t)\right)I_j(x(t_j)).\end{aligned}$$

(3.3)

Combine (3.1)-(3.3) and Corollary 2.1, we obtain

$$\left|\frac{d}{dt}(Tx)(t)\right| = \sup_{t\in R}\left|(T_j x)'(t)\right|$$

$$\leq \int_t^{t+T}\left|\lambda c(s) f(s,x(s),x(s-\tau(s)))\right|\left|-q(s)G(t,s)+\frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv-1}\right|ds$$

$$+\sum_{j=1}^{p}\left|-q(s)G(t,s)+\frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv-1}\right|\left|J_j(x(t_j))\right|$$

$$+\sum_{j=1}^{p}\left(\left|-q(t_j)p(t_j)G(t,t_j)\right|+\left|\frac{\exp\int_t^{t_j} p(v)dv}{\exp\int_0^T p(v)dv-1}p(t_j)\right|+\left|\frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv-1}q(t)\right|\right)\left|I_j(x(t_j))\right|$$

$$\leq\left(\lambda C\int_t^{t+T}\left|f(s,x(s),x(s-\tau(s)))\right|+\sum_{j=1}^{p}\left|J_j(x(t_j))\right|+\sum_{j=1}^{p}\left|I_j(x(t_j))\right|\left|p(t_j)\right|\right)ds\left((M\left\|Q\right\|+\frac{e^m}{e^l-1})\right)$$

$$+\sum_{j=1}^{p}\left|\frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv-1}q(t)\right|\left|I_j(x(t_j))\right|$$

$$\leq \lambda C(TW+U+PU)(M\left\|Q\right\|+\frac{e^m}{e^l-1})+\frac{e^m}{e^l-1}\left\|Q\right\|U,$$

where $\left\|Q\right\| = \max_{0\leq t\leq T}\left|q(t)\right|$, $\left\|P\right\| = \max_{0\leq t\leq T}\left|p(t)\right|$.

Hence $\{Tx : x\in K, \left\|x\right\|\leq\mu\}$ is a family of uniformly bounded and equicontinuous functions on $[0,T]$. By a theorem of Ascoli-Arzela, the function T is completely continuous.

Theorem 3.1. Suppose that (H1)-(H3), (2.1) and (2.5) and that there are positive constants R_1 and R_2 with $R_1 < R_2$ such that

$$\sup_{\left\|\phi\right\|=R_1,\phi\in K}\int_0^T\left|f(s,\phi(s),\phi(s-\tau(s)))\right|ds = P_1, \tag{3.4}$$

$$\sup_{\left\|\phi\right\|=R_1,\phi\in K}\left|I_j(\phi(t_j))\right| = I_1,$$

and

$$\inf_{\left\|\phi\right\|=R_2,\phi\in K}\int_0^T\left|f(s,\phi(s),\phi(s-\tau(s)))\right|ds = P_2, \tag{3.5}$$

$$\inf_{\left\|\phi\right\|=R_2,\phi\in K}\left|I_j(\phi(t_j))\right| = I_2,$$

for each λ satisfy

$$\frac{R_2}{MCP_2} < \lambda < \frac{R_1}{MCP_1}. \tag{3.6}$$

Then (1.2) has a positive T -periodic solution x with $R_1 \le \|x\| \le R_2$.

Proof. Let $x \in K$ and $\|x\| = R_1$. By (3.4) and (3.6), we have

$$\begin{aligned}|(Tx)(t)| &\le M \int_t^{t+T} |\lambda c(s) f(s, x(s), x(s-\tau(s)))| ds + M \sum_{j:t_j \in [t,t+T]} |I_j(x(t_j))| \\ &\le \lambda M C \int_t^{t+T} |f(s, x(s), x(s-\tau(s)))| ds + M \sum_{j:t_j \in [t,t+T]} |I_j(x(t_j))| \\ &< \frac{R_1}{MCP_1} MCP_1 + MpI_1 = R_1\end{aligned}$$

for all $t \in [0,T]$. This implies that $\|Tx\| \le \|x\|$ for $x \in K \cap \partial\Omega_1, \Omega_1 = \{x \in X, \|x\| < R_1\}$.

If $x \in K$ and $\|x\| = R_2$. By (3.5) and (3.6), we have

$$\begin{aligned}|(Tx)(t)| &\ge N \int_t^{t+T} |\lambda C(s) f(s, x(s), x(s-\tau(s)))| ds \\ &\ge \lambda NC \int_t^{t+T} |f(s, x(s), x(s-\tau(s)))| ds \\ &> \frac{R_2}{NCP_2} NC \int_t^{t+T} |f(s, x(s), x(s-\tau(s)))| ds \ge R_2\end{aligned}$$

for all $t \in [0,T]$. Thus, $\|Tx\| \ge \|x\|$ for $x \in K \cap \partial\Omega_2, \Omega_2 = \{x \in X, \|x\| < R_2\}$.

By Krasnoselskii's fixed point theorem, T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$. It is easy to say that (1.2) has a positive T -periodic solution x with $R_1 \le \|x\| \le R_2$. This completes the proof.

REFERENCES

[1]. R.P. Agarwal, D. O'Regan, Multiple nonnegative solutions for second order impulsive differential equations, Appl. Math. Comput. 114 (2000) 51–59.
[2]. F. Cong, Periodic solutions for second order differential equations, Appl. Math. Lett. 18 (2005) 957–961.
[3]. D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.
[4]. D. Jiang, On the existence of positive solutions to second order periodic BVPs, Acta Math. Sci. 18 (1998) 31–35.
[5]. D. Jiang, J. Wei, Monotone method for first- and second-order periodic boundary value problems and periodic solutions of functional differential equations, Nonlinear Anal. 50 (2002) 885–898.
[6]. S.G. Hristova, D.D. Bainov, Monotone-iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential-difference equations, J. Math. Anal. Appl. 1997 (1996) 1–13.
[7]. X. Lin, D. Jiang, Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, J. Math. Anal. Appl. 321 (2006) 501–514.
[8]. V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[9]. E.K. Lee, Y.H. Lee, Multiple positive solutions of singular two point boundary value problems for second order impulsive differential equation, Appl. Math. Comput. 158 (2004) 745–759.
[10]. Z. Zhang, J. Wang, On existence and multiplicity of positive solutions to periodic boundary value problems for singular nonlinear second order differential equations, J. Math. Anal. Appl. 281 (2003) 99–107.
[11]. D.R.Smart, Fixed Points Theorems, Cambridge University Press, Cambridge,1980.
[12]. Y. Liu, W. Ge, Positive solutions Positive solutions for nonlinear Duffing equations with delay and variable coefficients, Tamsui Oxf.J.Math. sci.20(2004)235-255.
[13]. Y. Wang, H. Lian, W. Ge, Periodic solutions for a second order nonlinear functional differential equation, Applied Mathematics letters, (2006)110-115.
[14]. Z. Wei, Periodic boundary value problem for second order impulsive integrodifferential equations of Mixed type in Banach space, J. Math. Appl. Anal. 195 (1995) 214–229.