International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321- 4767 P-ISSN: 2321-4759
www.ijmsi.org Volume 8 Issue 9 || September, 2020 || PP 35-45

New Exponential Extension Poisson Distribution: Properties and Applications

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ABSTRACT: *In this article, we have introduced a three-parameter lifetime model having monotonically increasing or decreasing and constant failure rate called the new exponential extension Poisson distribution. Various statistical and mathematical properties of the proposed model are discussed and calculated. The parameters of the proposed model are estimated by using the MLE method and also constructed the asymptotic confidence intervals and standard errors. Further, the Fisher information matrix is derived analytically to obtain the variance-covariance matrix for MLEs. The proposed methodology is illustrated using two real data sets. All the computations are preformed in R software. The potentiality of the proposed distribution is illustrated by using some graphical methods and statistical tests, where the proposed distribution provided a better fit and more flexible in comparison with some other lifetime distributions.*

KEYWORDS: *Exponential extension, Poisson, and Estimation.*

Date of Submission: 15-12-2020 Date of Acceptance: 30-12-2020

I. INTRODUCTION

In survival analysis, the length of the life of a component or device or a system is explained by lifetime distributions. The lifetime distributions are most frequently used in the field like life sciences, biological sciences, engineering and manufacturing, etc. For the analysis of survival data, many well-known probability models such as exponential, Weibull, Cauchy, gamma, etc. are used in many statistical kinds of literature. The exponential distribution is the most frequently used distribution due to the existence of simple elegant closed-form solutions to various survival analysis problems. The failure rate of the exponential distribution is stable but in actual practice, the failure rates are not always stable. Hence in several situations, it seems to be inadequate and unrealistic. For this, some modifications are needed to make exponential distribution more flexible. In recent, a new class of models has been introduced based on the modification of exponential distribution.

Gupta and Kundu (1999) have introduced the generalized exponential (GE) distribution, this extended family can accommodate data with increasing and decreasing failure rate functions, Kus (2007) has introduced the two-parameter exponential Poisson (EP) distribution by compounding exponential distribution with zero truncated Poisson distribution with decreasing failure rate. The CDF of PE distribution is,

$$F_{EP}(x) = \frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda\left(1-e^{-\beta x}\right)\right\}\right] \quad ; x>0, (\beta,\lambda)>0$$

While Barreto-Souza and Cribari-Neto (2009) have introduced generalized EP distribution having the decreasing or increasing or upside-down bathtub shaped failure rate. This is the generalization of the distribution proposed by Kus (2007) adding a power parameter to this distribution. Following the same fashion Cancho (2011) has introduced a new distribution family also based on the exponential distribution with an increasing failure rate function known as Poisson exponential (PE) distribution. The CDF of PE distribution can be expressed as

$$F_{PE}(x) = 1-\frac{1-\exp\left\{-\theta\left(1-e^{-\lambda x}\right)\right\}}{\left(1-e^{-\lambda}\right)} \quad ; x>0, (\lambda,\theta)>0$$

Louzada-Neto et al., (2011) has introduced a two-parameter Poisson-exponential with increasing failure rate by using the same approach as used by (Cancho, 2011) under the Bayesian approach. Alkarni and Oraby (2012) have presented a new lifetime class with a decreasing failure rate which is obtained by compounding truncated Poisson distribution and a lifetime distribution. The CDF of the Poisson family is given by,

$$F\left(x;\lambda,\underline{\theta}\right)=1-\frac{1-\exp\left\{-\lambda G\left(y,\underline{\theta}\right)\right\}}{\left(1-e^{-\lambda}\right)} \quad ; \ \lambda>0$$

Where $\underline{\theta}$ the parameter is space and $G\left(y,\underline{\theta}\right)$ is the CDF of baseline distribution. Using a similar approach the Weibull power series class of distributions with Poisson has presented by (Morais & Barreto-Souza, 2011). Mahmoudi and Sepahdar (2013) have presented a new four-parameter distribution with increasing, decreasing, bathtub-shaped, and unimodal failure rate called as the exponentiated Weibull–Poisson (EWP) distribution which has obtained by compounding exponentiated Weibull (EW) and Poisson distributions. The new compounding distribution named the Weibull–Poisson distribution was introduced by (Lu & Shi, 2012) having the shape of decreasing, increasing, upside-down bathtub-shaped or unimodal failure rate function. Furthur Kaviayarasu and Fawaz (2017) have made an extensive study on Weibull–Poisson distribution through a reliability sampling plan. Kyurkchiev et al. (2018) has used the exponentiated exponential-Poisson as the software reliability model. Louzada et al. (2020) has used different estimation methods to estimate the parameter of exponential-Poisson distribution using rainfall and aircraft data. Chaudhary and Kumar (2020) have presented the half logistic exponential extension distribution that can have the shape of decreasing, increasing, and upside-down bathtub-shaped failure rate function.

In this study, we propose a new distribution based on a new exponential extension (NEE) (Joshi, 2015) having monotonically increasing or constant failure rate function. The hazard rate function of NEE distribution is,

$$h\left(x\right)=\alpha\left(1+\frac{\beta}{x}\right)e^{-\beta/x} \quad ; x>0, \alpha>0, \beta>0$$

The motivation of this study is to obtain a more flexible model by adding just one extra parameter to the new exponential extension distribution to achieve a better fit to the real data. The different sections of this study are arranged as follows; in Section II we present the new exponential extension Poisson distribution with its properties. We extensively discussed the maximum likelihood estimation method in Section III. In Section IV using a real dataset, we present the estimated values of the model parameters and their corresponding asymptotic confidence intervals and fisher information matrix. Also, we present the different test criteria to assess the potentiality of the proposed model. Some concluding remarks are presented in Section V.

II. THE NEW EXPONENTIAL EXTENSION POISSON (NEEP) DISTRIBUTION

Let $G(x)$ and $g(x)$ be the baseline CDF and PDF respectively. The Poisson family defined by (Alkarni & Oraby, 2012) whose CDF and PDF respectively can be written as

$$F\left(x\right)=1-\frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda\left(1-G\left(x\right)\right)\right\}\right] \quad ; x>0, \lambda>0 \tag{2.1}$$

and

$$f\left(x\right)=\frac{1}{\left(1-e^{-\lambda}\right)}\lambda g\left(x\right)\exp\left\{-\lambda\left(1-G\left(x\right)\right)\right\} \quad ; x>0, \lambda>0 \tag{2.2}$$

In this study we have taken the new exponential extension (NEE) (Joshi, 2015) as baseline distribution with CDF and PDF respectively as follows,

$$G\left(x\right)=1-\exp\left\{-\alpha\, x\, e^{-\beta/x}\right\} \quad ; x>0, \alpha>0, \beta>0 \tag{2.3}$$

$$g\left(x\right)=\alpha\left(1+\frac{\beta}{x}\right)e^{-\beta/x}\exp\left\{-\alpha\, x\, e^{-\beta/x}\right\} \quad ; x>0, \alpha>0, \beta>0 \tag{2.4}$$

Hence, we can define the new Exponential Extension Poisson (NEEP) distribution as

Let X be a non negative random variable representing the lifetime of an item or component or a system in some population. The random variable X is said to follow the NEEP distribution with parameters $\left(\alpha,\beta,\lambda\right)>0$ if its cumulative distribution function is given by

$$F(x) = \frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda\exp\left(-\alpha\, x\, e^{-\beta/x}\right)\right\}\right] \quad ; x>0, (\alpha,\beta,\lambda)>0 \tag{2.5}$$

And its corresponding probability density function is

$$f(x) = \frac{\alpha\lambda}{\left(1-e^{-\lambda}\right)}\left(1+\frac{\beta}{x}\right)\exp\left\{-\frac{\beta}{x}-\alpha\, x\, e^{-\beta/x}-\lambda\exp\left(-\alpha\, x\, e^{-\beta/x}\right)\right\} \quad ; x>0. \tag{2.6}$$

Figure 1 exhibits the graph for PDF and hazard function for NEEP distribution for different values of parameters. From Figure 1 (left panel), the density function of the NEEP distribution can bear different shapes according to the values of the parameters. Figure 1 (right panel) demonstrates the increasing, decreasing, and constant graph of the hazard function.

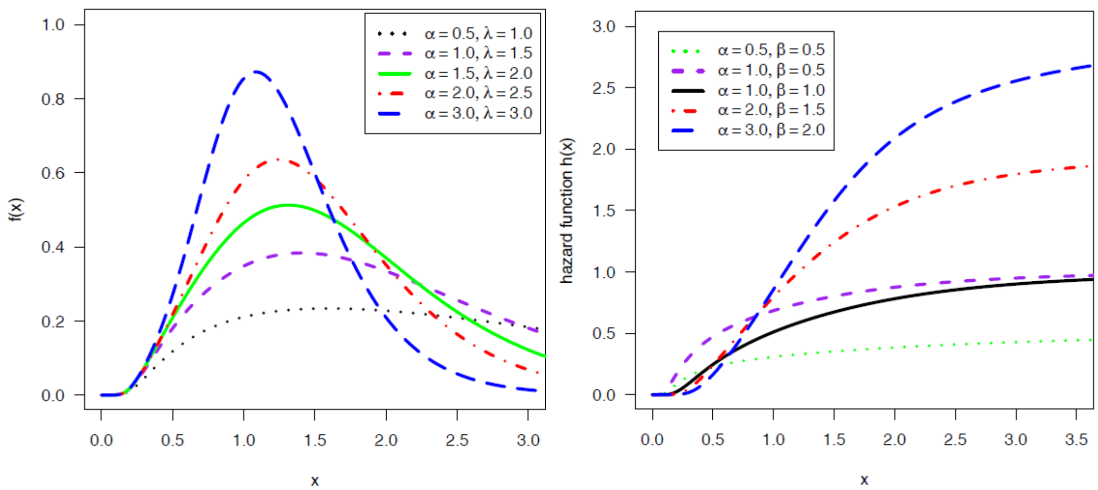

Figure 1. Graph of PDF (left panel) and hazard function (right panel) for different values of the parameters.

Survival function:

The survival function $R(t)$, which is the probability of an item not failing up to time *t*, is defined by $R(t) = 1 - F(t)$. The survival /reliability function of a new exponential extension Poisson distribution is given by

$$R(t) = \frac{1}{\left(1-e^{-\lambda}\right)}\left[\exp\left\{-\lambda\exp\left(-\alpha\, t\, e^{-\beta/t}\right)\right\}-\exp(-\lambda)\right] \quad ; t>0, (\alpha,\beta,\lambda)>0 \tag{2.7}$$

The hazard rate function (HRF)

Let t be the lifetime of a component or item and the probability that it will not survive for an additional time φt then, hazard rate function is,

$$h(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1-F(t)}; \; 0<t<\infty$$

where R(t) is a reliability function.

Hence let, $X \sim NEEP(\alpha,\beta,\lambda)$ then its hazard rate function is

$$h(x) = \frac{\alpha\lambda\left(1+\frac{\beta}{x}\right)\exp\left\{-\frac{\beta}{x}-\alpha\, x\, e^{-\beta/x}-\lambda\exp\left(-\alpha\, x\, e^{-\beta/x}\right)\right\}}{\left[\exp\left\{-\lambda\exp\left(-\alpha\, x\, e^{-\beta/x}\right)\right\}-\exp(-\lambda)\right]}; x>0 \tag{2.8}$$

The quantile function of NEEP distribution is,
The value of the p[th] quantile can be obtained by solving the following equation,

$$Q(p) = F^{-1}(p)$$

And we get the quantile function by inverting (2.3) as

$$\alpha\, x e^{-\beta/x} + \log c = 0 \quad \text{where } c = -\frac{1}{\lambda}\log\left[1-\left(1-e^{-\lambda}\right)(1-p)\right] \tag{2.9}$$

For the generation of the random numbers of the NEEP distribution, we suppose simulating values of random variable X with the CDF (2.5). Let U denote a uniform random variable in (0, 1), then the simulated values of X can be obtained by

$$\alpha\, x e^{-\beta/x} + \log c = 0 \quad \text{where } c = -\frac{1}{\lambda}\log\left[1-\left(1-e^{-\lambda}\right)(1-u)\right] \tag{2.10}$$

Skewness and Kurtosis:
The Bowley's skewness based on quartiles is,

$S_{kb} = \dfrac{Q_3 + Q_1 - 2Median}{Q_3 - Q_1}$, where and $Q_3 \; and \; Q_1$ are the upper quartile and lower quartile respectively.

The coefficient of kurtosis based on octiles given by (Moors, 1988) is

$$K_u(Moors) = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(3/4) - Q(1/4)}$$

III. MAXIMUM LIKELIHOOD ESTIMATION METHOD

If $x_1, x_2, \ldots, x_n$ is a random sample from $NEEP(\alpha, \beta, \lambda)$ then the likelihood function, $L(\alpha, \beta, \lambda)$ is given by,

$$L(\delta; x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n / \delta) = \prod_{i=1}^{n} g(x_i / \delta)$$

$$L(\alpha,\beta,\lambda) = \frac{\alpha\lambda}{\left(1-e^{-\lambda}\right)} \prod_{i=1}^{n} \left(1+\frac{\beta}{x_i}\right) \exp\left\{-\frac{\beta}{x_i} - \alpha\, x_i\, e^{-\beta/x_i} - \lambda \exp\left(-\alpha\, x_i\, e^{-\beta/x_i}\right)\right\}; \quad \alpha,\beta,\lambda>0,\; x>0$$

The log-likelihood density is

$$\ell = n\log(\alpha\lambda) - n\log\left(1-e^{-\lambda}\right) + \sum_{i=1}^{n}\log\left(1+\frac{\beta}{x_i}\right) - \sum_{i=1}^{n}\frac{\beta}{x_i} - \alpha\sum_{i=1}^{n} x_i\, e^{-\beta/x_i} - \lambda\sum_{i=1}^{n}\exp\left(-\alpha\, x_i\, e^{-\beta/x_i}\right) \tag{3.1}$$

Differentiating (3.1) with respect to α, β, and λ we get,

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i\, e^{-\beta/x_i} + \lambda\sum_{i=1}^{n} x_i\, e^{-\beta/x_i - \alpha\, x_i e^{-\beta/x_i}}$$

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{n}\frac{1}{\beta + x_i} - \sum_{i=1}^{n}\frac{1}{x_i} + \alpha\sum_{i=1}^{n} e^{-\beta/x_i} - \alpha\lambda\sum_{i=1}^{n} e^{-\beta/x_i - \alpha\, x_i e^{-\beta/x_i}}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \frac{n}{e^{\lambda}-1} - \sum_{i=1}^{n} e^{-\alpha\, x_i e^{-\beta/x_i}}$$

Solving non-linear equations $\dfrac{\partial l}{\partial \alpha} = 0,\ \dfrac{\partial l}{\partial \beta} = 0 \; and \; \dfrac{\partial l}{\partial \lambda} = 0$, for α, β, and λ, we get the maximum likelihood estimate $\hat{\alpha}, \hat{\beta} \; and \; \hat{\lambda}$ of the parameters α, β, and λ. The maximization of (3.1) can be obtained by using computer software like R, Matlab, etc. For the interval estimation of α, β, and λ and testing of the hypothesis, we have to calculate the observed information matrix. The observed information matrix for α, β, and λ can be obtained as

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\beta} & M_{\alpha\lambda} \\ M_{\beta\alpha} & M_{\beta\beta} & M_{\beta\lambda} \\ M_{\lambda\alpha} & M_{\lambda\beta} & M_{\lambda\lambda} \end{bmatrix}$$

where

$$M_{\alpha\alpha} = -\frac{n}{\alpha^2} + \lambda \sum_{i=1}^{n} x_i^2 e^{-2\beta/x_i - \alpha x_i e^{-\beta/x_i}}$$

$$M_{\beta\beta} = -\frac{1}{(\beta + x_i)^2} - \alpha \sum_{i=1}^{n} \frac{e^{-\beta/x_i}}{x_i} - \alpha\lambda \sum_{i=1}^{n} \left(\alpha e^{-\beta/x_i} - \frac{1}{x_i} \right) e^{-\beta/x_i - \alpha x_i e^{-\beta/x_i}}$$

$$M_{\lambda\lambda} = -\frac{n}{\lambda^2} - \frac{n e^{\lambda}}{\left(e^{\lambda} - 1\right)^2}$$

$$M_{\alpha\beta} = \sum_{i=1}^{n} x_i^2 e^{-\beta x_i} - \lambda \sum_{i=1}^{n} x_i \left(-x_i + \alpha e^{-\beta/x_i} \right) e^{-\beta/x_i - \alpha x_i e^{-\beta/x_i}}$$

$$M_{\alpha\lambda} = \sum_{i=1}^{n} x_i e^{-\beta x_i - \alpha x_i e^{-\beta/x_i}} \text{ and } M_{\beta\lambda} = \alpha \sum_{i=1}^{n} e^{-\beta x_i - \alpha x_i e^{-\beta/x_i}}$$

Let $\Phi = (\alpha, \beta, \lambda)$ denote the parameter space and the corresponding MLE of Φ is $\hat{\Phi} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ as then $\left(\hat{\Phi} - \Phi\right) \rightarrow N_3\left[0, \left(M(\Phi)\right)^{-1}\right]$ follows the asymptotic multivariate normal distribution, where $M(\Phi)$ is Fisher's information matrix. By applying the Newton-Raphson algorithm to maximize the likelihood (3.1) produces the observed information matrix and hence the variance-covariance matrix is obtained as,

$$\left[M(\Phi)\right]^{-1} = \begin{bmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{cov}(\hat{\alpha}, \hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\beta}, \hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha}, \hat{\lambda}) & \mathrm{cov}(\hat{\beta}, \hat{\lambda}) & \mathrm{var}(\hat{\lambda}) \end{bmatrix} \tag{3.2}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, β, and λ can be constructed as,

$$\hat{\alpha} \pm Z_{\alpha/2} SE(\hat{\alpha}) \;,\; \hat{\beta} \pm Z_{\alpha/2} SE(\hat{\beta}) \text{ and } \hat{\lambda} \pm Z_{\alpha/2} SE(\hat{\lambda}) \tag{3.3}$$

where $Z_{\alpha/2}$ is the upper percentile of standard normal variate.

IV. ILLUSTRATION WITH REAL DATASETS

In this section, we have presented the applicability of new exponential extension Poisson distribution using two real datasets used by earlier researchers. To compare the potentiality of the proposed model, we have considered the following four distributions.

a. Exponentiated Exponential Poisson (EEP):

The probability density function of EEP (Ristić & Nadarajah, 2014) can be expressed as

$$f_{EEP}(x) = \frac{\alpha\beta\lambda}{\left(1 - e^{-\lambda}\right)} e^{-\beta x} \left(1 - e^{-\beta x}\right)^{\alpha - 1} \exp\left\{-\lambda\left(1 - e^{-\beta x}\right)^{\alpha}\right\} \quad ; x > 0, \alpha > 0, \lambda > 0$$

b. The new exponential extension (NEE)

The probability density function of NEE (Joshi, 2015) is

$$f_{NEP}(x) = \alpha\left(1 + \frac{\beta}{x}\right) e^{-\beta/x} \exp\left\{-\alpha x e^{-\beta/x}\right\} \quad ; x > 0, \alpha > 0, \beta > 0$$

c. Power Lindley distribution (LP):

The probability density function of power Lindley distribution (Ghitany et al., 2013) with parameters α and β is

$$f_{PL}(x) = \frac{\alpha\beta^{2}}{\beta+1}\left(1+x^{\alpha}\right)x^{\alpha-1}e^{-\beta x^{\alpha}} \quad ; x \geq 0, \alpha > 0, \beta > 0.$$

d. Weibull distribution:

The probability density function of Weibull (W) distribution is

$$f_{w}(x) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-(x/\lambda)^{\theta}}; \ \lambda\theta > 0, x \geq 0$$

Dataset-I

The data set is originally considered by (Bader & Priest, 1982). The data given represent the strength measured in GPA for single carbon fibers of 10mm in gauge lengths with sample size 63 and they are as follows:
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

In Figure 2 we have displayed the graph of profile log-likelihood functions of ML estimates of α, β, and λ. We have found that ML estimates of α, β, and λ exist and can be obtained uniquely.

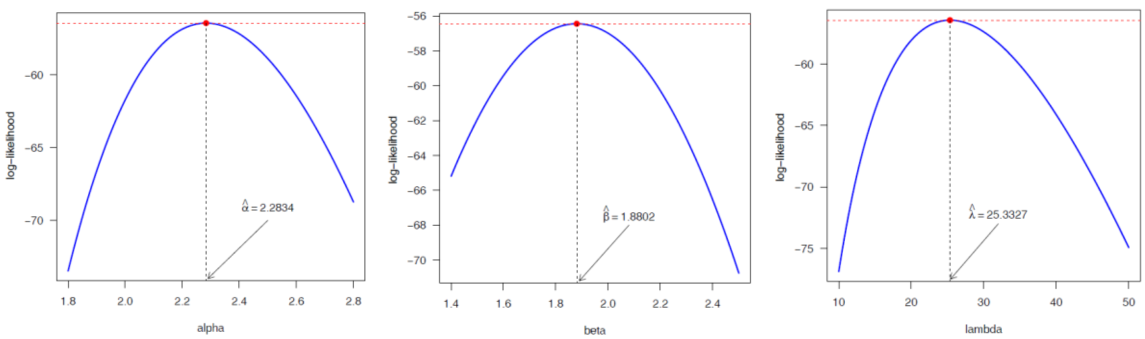

Figure 2. The plots of the profile log-likelihood function of ML estimates of α, β, and λ.

We have calculated the MLEs directly by using the optim() function (Schmuller, 2017) in R software (R Core Team, 2020) by maximizing the likelihood function (3.1). We have obtained $\hat{\alpha}$ = 2.2834, $\hat{\beta}$ = 1.8802, $\hat{\lambda}$ = 25.3327 and corresponding value of Log-Likelihood value is -56.4446. In Table 1 we have presented the MLE's with their standard errors (SE) and 95% confidence interval for α, β, and λ.

Table 1

MLE, SE and 95% confidence interval for α, β, and λ

Parameter	MLE	SE	95%ACI
alpha	2.2834	0.3849	(0.3849, 3.0378)
beta	1.8802	0.6280	(0.6648, 3.0952)
lambda	25.3327	7.9095	(9.8301, 40.8353)

An estimate of the variance-covariance matrix by using MLEs, using equation (3.2) is

$$\begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\lambda}) \\ cov(\hat{\alpha},\hat{\beta}) & var(\hat{\beta}) & cov(\hat{\lambda},\hat{\beta}) \\ cov(\hat{\alpha},\hat{\lambda}) & cov(\hat{\lambda},\hat{\beta}) & var(\hat{\lambda}) \end{pmatrix} = \begin{pmatrix} 0.1481145 & 0.2152734 & -1.105258 \\ 0.2152734 & 0.3944355 & -3.533065 \\ -1.105258 & -3.533065 & 62.560942 \end{pmatrix}$$

To evaluate the goodness of fit of a given distribution we generally use the PDF and CDF plot. To get the additional information we have to plot Q-Q and P-P plots. In particular, the Q-Q plot may provide information about the lack-of-fit at the tails of the distribution, whereas the KS plot emphasizes the lack-of-fit. From Figure 3 we have shown that the NEEP model fits the data very well.

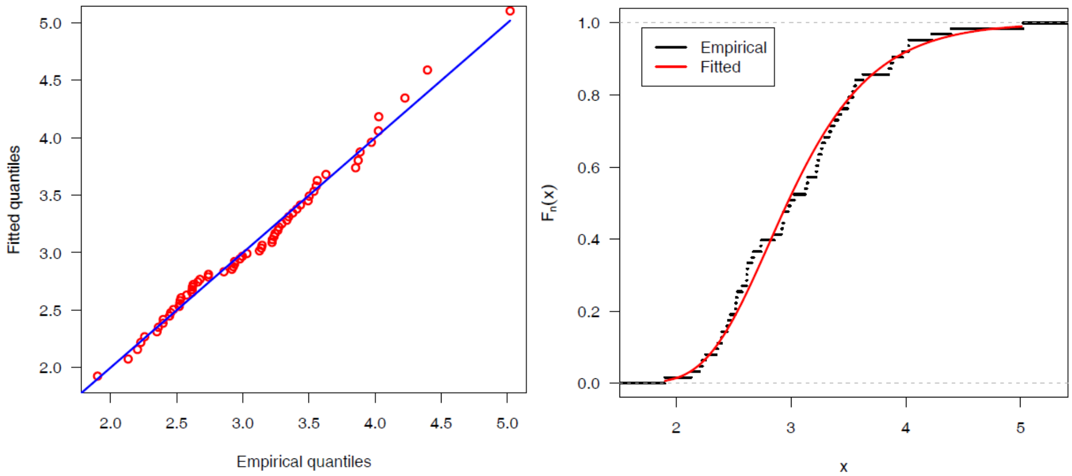

Figure 3. The Q-Q plot (left panel) and KS plot (right panel) of NEEP distribution

For the assessment of the potentiality of the NEEP distribution, we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) and these are presented in Table 2.

Table 2
Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
PEE	56.4446	118.8892	125.3186	119.2960	121.4179
EEP	57.0630	120.1261	126.5555	120.5328	122.6548
NEE	58.2218	120.4435	124.7298	120.6435	122.1293
LP	59.8601	123.7203	128.0066	123.9203	125.4061
Weibull	61.9570	127.9140	132.2002	128.1140	129.5998

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of NEEP, Exponentiated Exponential Poisson (EEP), new exponential extension (NEE), Power Lindley (LP) and Weibull distributions are presented in Figure 4.

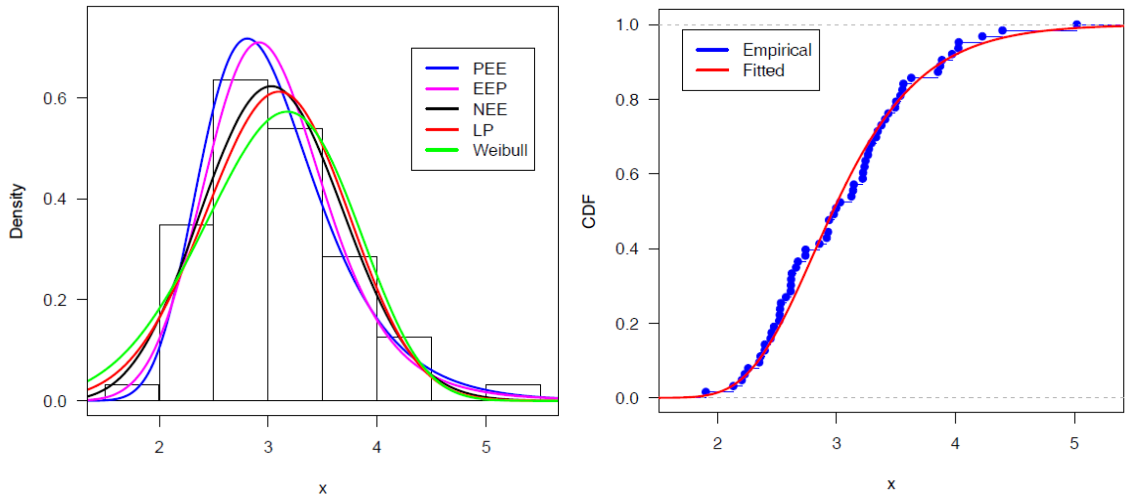

Figure 4. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the NEEP distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics. These three statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. From Table 3 the result shows that the NEEP distribution has the minimum value of the test statistic and higher *p*-value, hence we conclude that the NEEP distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 3
The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
PEE	0.0859(0.7413)	0.0687(0.7614)	0.3551(0.8914)
EEP	0.0907(0.6784)	0.0714(0.7451)	0.4002(0.8480)
NEE	0.0864(0.7352)	0.0670(0.7717)	0.4781(0.7686)
LP	0.0896(0.6929)	0.0908(0.6337)	0.6587(0.5937)
Weibull	0.0876(0.7191)	0.1242(0.4798)	0.9330(0.3941)

Dataset-II

The following dataset represents the waiting time (in minute) of 100 bank customers (Ghitany et al., 2008).
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5

We have obtained the MLEs $\hat{\alpha} = 0.1625$, $\hat{\beta} = 1.9239$, $\hat{\lambda} = 1.2806$ and corresponding value of Log-Likelihood value is -316.9819. In Table 4 we have presented the MLE's with their standard errors (SE) for α, β, and λ.

Table 4
MLE and SE for α, β, and λ

Parameter	MLE	SE
alpha	0.1625	0.0250
beta	1.9239	1.0762
lambda	1.2806	0.9717

An estimate of the variance-covariance matrix by using MLEs, using equation (3.2) is

$$\begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\lambda}) \\ cov(\hat{\alpha},\hat{\beta}) & var(\hat{\beta}) & cov(\hat{\lambda},\hat{\beta}) \\ cov(\hat{\alpha},\hat{\lambda}) & cov(\hat{\lambda},\hat{\beta}) & var(\hat{\lambda}) \end{pmatrix} = \begin{pmatrix} 0.00063 & -0.0101 & 0.01850 \\ -0.0101 & 1.1581 & 0.8150 \\ 0.01850 & 0.8150 & 0.9441 \end{pmatrix}$$

In Figure 5 we have displayed the graph of profile log-likelihood functions of ML estimates of α, β and λ. We have found that ML estimates of α, β, and λ exist and can be obtained uniquely.

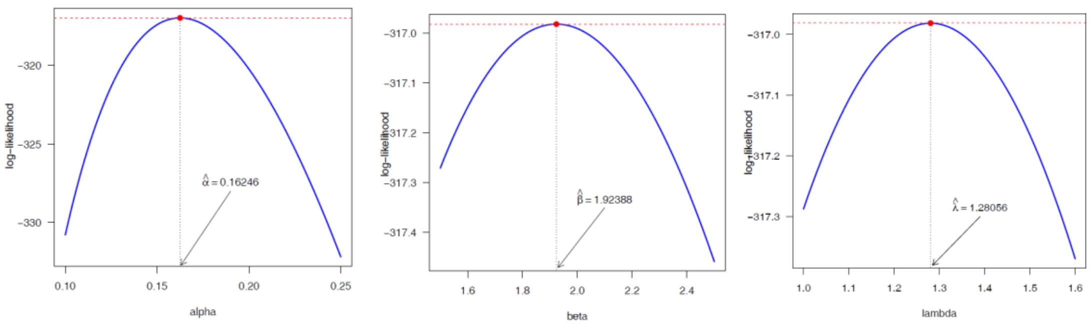

Figure 5. The plots of the profile log-likelihood function of ML estimates of α, β, and λ.

To evaluate the goodness of fit of a given distribution we generally use the PDF and CDF plot. To get the additional information we have to plot Q-Q and P-P plots. In particular, the Q-Q plot may provide information about the lack-of-fit at the tails of the distribution, whereas the KS plot emphasizes the lack-of-fit. From Figure 6 we have shown that the NEEP model fits the data very well.

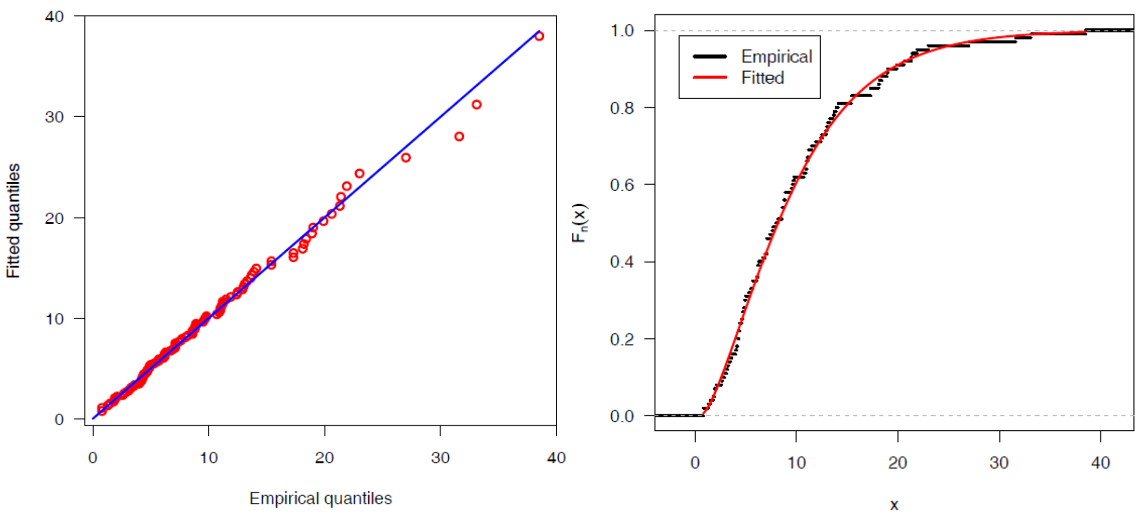

Figure 6. The Q-Q plot (left panel) and KS plot (right panel) of NEEP distribution

For the assessment of the potentiality of the NEEP distribution, we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) and these are presented in Table 5.

Table 5
Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
PEE	316.9819	639.9638	647.7793	640.2138	643.1269
EEP	317.0196	640.0393	647.8548	640.2893	643.2024

NEE	317.8478	639.6955	644.9058	639.8192	641.8042
LP	318.3186	640.6372	645.8475	640.7609	642.7459
Weibull	318.7307	641.4614	646.6717	641.5851	643.5701

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of NEEP, Exponentiated Exponential Poisson (EEP), new exponential extension (NEE), Power Lindley (LP) and Weibull distributions are presented in Figure 7.

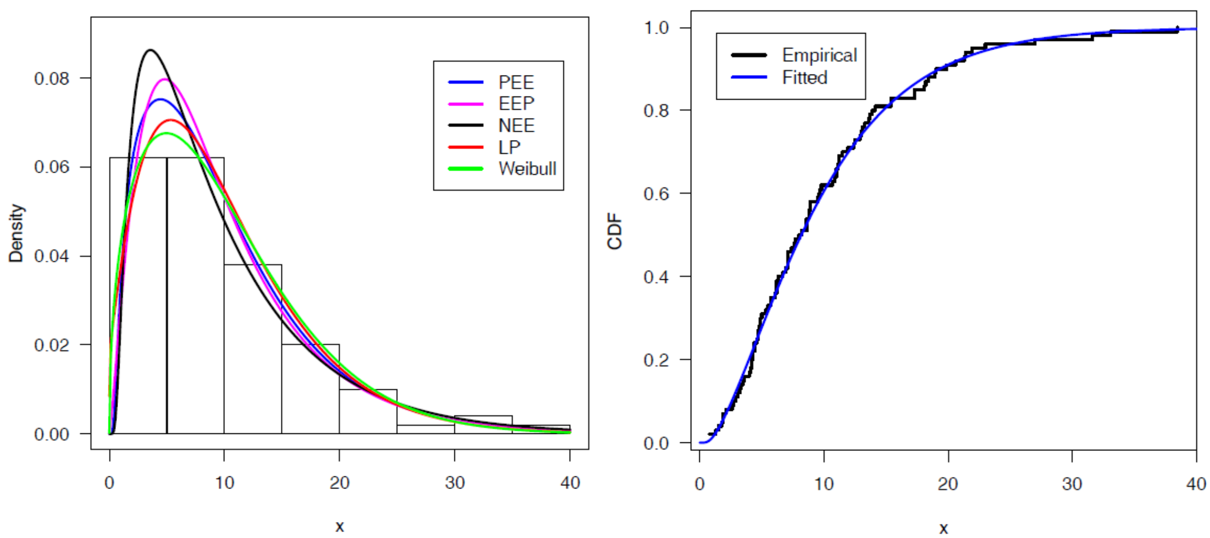

Figure 7. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the NEEP distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics. These three statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. From Table 6 the result shows that the NEEP distribution has the minimum value of the test statistic and higher *p*-value, hence we conclude that the NEEP distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 6
The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
PEE	0.0430(0.9926)	0.0225(0.9942)	0.1639(0.9973)
EEP	0.0366(0.9993)	0.0173(0.9989)	0.1259(0.9997)
NEE	0.0639(0.8093)	0.0520(0.8650)	0.3465(0.8993)
LP	0.0520(0.9498)	0.0458(0.9025)	0.3028(0.9359)
Weibull	0.0578(0.8920)	0.0611(0.8084)	0.4058(0.8426)

V. SUMMARY AND CONCLUSION

In this study, we have introduced a three-parameter probability distribution called new exponential extension Poisson distribution. A comprehensive study of some statistical and mathematical properties of the proposed distribution including the derivation of explicit expressions for its reliability function, survival function, hazard function, the quantile function which is useful for calculating partition values and skewness and kurtosis, also we have presented skewness and kurtosis, and simulation of random numbers from the proposed distribution. The unknown model parameters are estimated using the method of maximum likelihood estimation and constructed their corresponding confidence intervals. The graph of the PDF of the proposed distribution has shown that its shape is the skewed model and flexible for modeling real-life data. Also, the graph of the hazard function is monotonically decreasing or increasing according to the value of the model parameters. The performance of the proposed distribution has been evaluated by considering two real-life datasets and the results revealed that the proposed distribution is much flexible as compared to some other fitted distributions.

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Ramesh Kumar Joshi, et. al. "New Exponential Extension Poisson Distribution: Properties and Applications." *International Journal of Mathematics and Statistics Invention (IJMSI)*, vol. 08(09), 2020, pp. 35-45.