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Discontinuous Finite Element Adaptive Methods for Biharmonic Eigenvalue Problems with Simply Supported Boundary Conditions

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ABSTRACT*: Biharmonic eigenvalue equation is a typical fourth-order partial differential equation, which is an important partial differential equation model in elastic thin plate, biophysics and other fields, and its efficient numerical solution has been a hot spot and difficulty in related fields. The discontinuous finite element method has high plasticity and adaptability, and has become an important numerical method for solving various kinds of partial differential equations and practical problems. In this paper, we use the discontinuous finite element method to study the eigenvalue problem of biharmonic equations with simply supported boundary conditions, and introduce a posterior error index based on residual through discontinuous Galerkin discretization, and obtain the complete posterior error estimation results of this method. The performance of this index is verified in an adaptive mesh refiner.*
KEYWORDS*: Biharmonic eigenvalue equation, Discontinuous Galerkin method, Posterior error, adaptive.*

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I. INTRODUCTION

The biharmonic equation originates from the elastic thin plate theory in the field of continuum mechanics. The fourth-order boundary value problem is a kind of special boundary value problem of partial differential equations, which often appears in thin plate theory of elasticity, phase field model and mathematical biology, which makes biharmonic equations widely used. Many scholars have also been committed to the numerical solution of biharmonic equations, and its solution methods are constantly optimized and innovative. The finite difference method was used to solve biharmonic equations[1]. Liu used the mixed finite element method to solve the biharmonic equation[2], that is, by introducing intermediate variables, the biharmonic equation was reduced to two second-order equations, and the mixed finite element space satisfying certain conditions was used to discretize corresponding mixed variational problem, so as to obtain the numerical solutions of the original variables and intermediate variables satisfying the original equation. Discontinuous Galerkin finite element method is a kind of finite element method using completely discontinuous basis function, which can solve more complex boundary problems, and is easy to realize the selection of local mesh and each element polynomial. Therefore, discontinuous Galerkin method is often used to solve various eigenvalue problems, such as Steklov eigenvalue problem, Laplacian eigenvalue problem, biharmonic eigenvalue problem, etc. Emmanuil derived the DG scheme of the biharmonic equation[3]. The internal penalty discontinuous finite element method is to penalty the jump of the approximating solution on the common edge or common surface of the element, which is more flexible than the finite element method. [4] constructed the hp internal penalty discontinuity Galerkin finite element method for biharmonic equations and analyzed the prior error of the method. In this paper, the biharmonic eigenvalue problem with simply supported boundary is studied by discontinuous finite element method in internal penalty discontinuous galerkin(IPDG) format, and a posterior error estimation is established to verify the reliability and validity of the posterior error estimation of the discontinuous finite element method. The results show that the adaptive algorithm can achieve the optimal convergence order.

II. BASIC THEORETICAL PREPARATION

to represent a standard Lebesgue space, where , The corresponding norm is expressed by. In this paper, the norm of is represented by We also use to express the standard Hilbert Sobolev space of real functions defined at with index and the corresponding norm and semi-norm areand. Let be the bounded open polygon region of , and let represent its boundary. Consider the simply supported boundary condition eigenvalue problem: find and , such that

Denote

and define a continuous bilinear form

Then, there exists two positive constants and independent of and , such that the bilinear form is satisfied

The weak form of (2.1) is to find such that

Let be a conforming subdivision of into disjoint triangular or quadrilateral elements , on this assumption that the subdivision is shape regular and constructed by affine mapping, where, with nonsingular Jacobin, where is the reference triangle or quadrilateral. It is assumed that the mapping is constructed to ensure that and the elemental edges are straight line segments.

The broken Laplacian is defined by

For a non-negative integer , is used to represent the set of all polynomials of degree at most if is a reference triangle, and is used to represent the set of polynomials of tensor product if is a reference quadrilateral. For , consider its finite element space

.

We use to represent the union (including the boundary) of all one-dimensional unit edges associated with the subdivision . In addition, we decompose into two disjoint subsets, i.e. , where.

Letand be two elements of the shared edge . Define the outward normal unit vectors on corresponding to and , respectively, as and . For functions and , these functions may be discontinuous in, the following is defined for,

, , ,.

If , then these definitions are changed as follows:

, , , .

With the above definition, it can be verified

To define , and collect them into the elementwise constant function, with , and . We always assume that the families of meshes considered are locally quasi-uniform, there are constants independent of , for any pair of elements and in , that share an edge, we have

.

We first introduce the lifting operator by

And the lifting operatorhas stability: for , there is

Where , .

Proof. See [5].

Define bilinear form as by

here the internal penalty parameter , of the segmentation constant is defined as

where , in order to guarantee the stability of the IPDG method defined in (2.7), , must be selectively large enough.

The finite element approximation of (2.4) is to find , such that

The source problem of (2.4) is to find , such that

The DG approximation of (2.10) is to find , such that

Define the linear bounded operator satisfying

The equivalent operator from of (2.4) is

By using (2.10), the corresponding discrete solution operator can be defined:

The equivalent operator from of (2.10) is

From the consistency of discontinuous finite element method, let be the solution of (2.12), and , then

From (2.11) and (2.16), we obtain

For any function , introduce sum space , that assigns a locally discontinuous finite element norm, where the energy norm is defined as follows:

There is is continuous and coercive

where is a piecewise continuous function, and are positive constants depending only on the mesh parameters.
Proof. For , using the Cauchy-Schwarz inequality, we have

Continuity is valid.
Next, we prove (2.20), using the definition of norm and the Young's inequality, we obtain

When , the proof is completed.
Let be the solution of (2.12), and , assuming the following regularity estimate holds:

Let be the quadratic interpolation of , then:

also .

Lemma 2.1. (Proposition 4.9 in [6]) Let and then there exists the polynomial , satisfying

Introduce the global interpolation operator such that , for the vector-value function define

Lemma 2.2. (lemma 2.1 in [7]) Let , , and , for any with , there exists a positive constant C independent of such that

Theorem 2.1. Let and be the solution of (2.10) and (2.11), for all , and , then there holds

Proof. Firstly, we prove (2.25) by utilizing (2.17), (2.19) and (2.20), we obtain

From lemma 2.2, the inverse estimate and the definition of energy norm, we deduce

Also

From the trace inequality, the definition of energy norm and (2.21), we deduce

Similarly,

Then

Using the triangle inequality, we get (2.25).
Next, we prove (2.26). By (2.18), let , having

can be estimated from (2.23):

can be estimated from (2.24), the trace inequality and the inverse estimate:

Similarly, we get :

Using (2.32), (2.33) and (2.34), we get

By using the error estimate and the interpolation estimate, we obtained

Then (2.26) directly from (2.25), (2.35) and (2.36), the proof is completed.

Theorem 2.2. Let and be the solution of (2.10) and (2.11), then there holds:

Proof. is the quadratic interpolation of , form (2.17) and (2.22), we have

From , we derive

From lemma 2.2, the inverse estimate, definition of energy norm, (2.21) and taking , we deduce

By the trace inequality with , the interpolation estimates and the definition of energy norm, we get

From the trace inequality, (2.21), (2.22) and the definition of energy norm, we derive

Then (2.37) directly from (2.39), (2.40), (2.41) and (2.43).
Next, we prove (2.38). From (2.26), (2.37) and (2.43), we get

So, (2.38) is valid.
Taking in (2.26), and the regularity estimate yields the following stable estimate:

Let be the th eigenvalue of (2.4), with algebraic multiplicities and the ascent ,where . When , eigenvalue of (2.9) will converge to . Let be the generalized eigenvector space of (2.4) related to , be the direct sum of the generalized eigenvector space of (2.9) related to , and converge to .
The following theorem can be proven using a similar method as proof Theorem 3.1 in reference [8].

Theorem 2.3. The following inequality holds

Let be the direct sum of the generalized eigenvector space of (2.9), with ,then there exists eigenvalue function of (2.4) such that

III. POSTERIOR ERROR ESTIMATION

i. Estimators of eigenfunctions and their reliability

Let be the eigenpair of (2.9), and define element residuals and surface residuals on each element and , respectively, as follows,

Define local error indicators on the of each unit

where .
The global error indicator is

Lemma 3.1. We assume that the mesh is constructed as above. Then there exists an operator that satisfies

with and being a constant that is independent of and.
Note that the recovery operator maps elements of onto a -conforming space consisting of macro-elements of degree 4.
Proof. See [3].

Theorem 3.1. Let and be the eigenpairs of (2.4) and (2.9), for any , the following formula holds

Proof. Let , with in lemma 3.1, then the error can be decomposed into

Since is the solution to the weak-form problem, we have , where . We have

Then

By , there is , then , and by in (3.5), there is

We have   is a linear approximation to , then is a constant independent of , , from [9] we get

By (3.7), then

By (2.5), (2.7), Green's formula and the definition of the lifting operator, there is

From the inverse estimate, the stability of the lifting operator, the trace inequality, (3.7) and Poincaré-Friedrichs inequalities, we get

Using , the triangle inequality and the stability of the lifting operator

Substituting (3.8), (3.9) and (3.10) into (3.6), and using the Cauchy-Schwarz inequality, we obtain

Then

Theorem 3.1 can be proved by Lemma 3.1, (3.11), (3.12) and the triangle inequality.

For the residual term , reference [3] shows that it does not affect the upper bound, and it can be seen from theorem 2.3 that when ascent , and are both small quantities of higher order . Therefore, it can be seen from (3.3) that the indicator of error estimation is one of the upper bounds of the discontinuous finite element energy norm, so the error estimation is reliable.

ii. Effectiveness of the eigenfunction estimator

Theorem 3.2. Under theorem 3.1, there is
for any ,

for any ,

for any ,

for any

for any

Proof. First prove Given that is a subspace of . Fix, and let , with , be a polynomial function on . Setting and taking as above in (3.4) yields

noting that on and that . We have

Letting , where is the standard internal bubble function (which is defined by , where are the barycentric coordinates of the reference triangle , then , and if is the reference rectangle, then . We have

Then applying (3.14), (3.15) and the Cauchy-Schwarz inequality yields

is valid.

For any , we have , which gives For any , we have , then we get

Next prove . For each inner edge , we define the largest diamond in as , where is the diagonal of the diamond . And we define the bubble function on the diamond . And there is an affine which has a value of 0 along edge e, i.e. . Thus is fully defined as a symbol, which is irrelevant to the discussion. The above definition gives the function , where , and on , where , then we have the following properties:

and along edge we have .

We set where is a constant function in the direction of normal, i.e., , and substitute and into (3.4), we deduce

Letting in (3.18), we derive

From scaling argument and norm equivalence, let , where represents the length of a line perpendicular to in intersecting at point , so there is

From (3.18) and (3.19), we have

Substitute (3.20) and into (3.21), by the Cauchy-Schwarz inequality, and multiply (3.21) by , so is proved.

Similarly, the same as the above, have

Letting , is defined the same as , and substitute and into (3.4), we have

Let into (3.23), there is

From the above, there are the following

The following can be obtained by (3.23) and (3.24)

By substituting and (3.25) into (3.26) and multiplying both sides of (3.26) by , is proved.

Theorem 3.3. Under Theorem 3.1 and theorem 3.2, we have

Proof. According to the definition of and theorem 3.2, (3.27) can be obtained, and using the definition of energy norm, (3.28) can be obtained.

Theorem 3.3 shows that the error estimation indicator is valid.

iii. The reliability of the estimators for the eigenvalues

Lemma 3.2. Let and be the eigenpairs of (2.4) and (2.9), respectively, then

Theorem 3.4. Under the condition of lemma 3.2, let , then

Proof. Theorem 2.3 shows that is a term higher than , so from lemma 3.1 and (3.3), we have

From lemma 2.2, the inverse estimate and the definition of energy norm, we deduce

From the trace inequality and the definition of energy norm, we derive

Substituting (3.32) and (3.33) into (3.31), and then from (3.3) and the Cauchy-Schwarz inequality, we get (3.30), that is, the proof is complete.

From theorem 3.1 and theorem 3.3, we know that the estimator of the eigenfunction error is reliable and efficient. Therefore, an adaptive algorithm based on this estimator indicator can generate a good gradient grid such that the approximate eigenfunction reaches the optimal convergence rate in . Thus, we expect:

Therefore, from (3.30), we get . Thus, can be regarded as the error estimation indicator of . The following numerical experiments show that as the error estimation indicator of is reliable and efficient.

IV. NUMBERICAL EXPERIMENTS

In this section, we report some numerical experiments to demonstrate the effectiveness of our approach. Considering the problem (2.1), our program is compiled under the iFEM package and we use the DG method where the penalty coefficient is to do the calculation. Consider the following two test domain: square domain with vertex of, hexagonal domains with vertex of   Since the exact eigenvalue is unknown, we take the reference eigenvalue in the square domain and the first two reference eigenvalues in the hexagon domain.

Table 1: Results of numerical solutions of quadratic eigenvalues for region ,with an initial grid of 1/8

	1	768	1.0e+02*4.804360813618129	90.7995813618128
	2	1056	1.0e+02*4.0162473161865	11.98823162
	4	1728	1.0e+02*3.92381659951859	2.74515995185873
	6	3888	1.0e+02*3.90616393229752	2.62161409558774
	8	8760	1.0e+02*3.92258114095587	0.979893229753657
	10	18564	1.0e+02*3.9012227109147	0.485771091471861
	12	42378	1.0e+02*3.89847285964222	0.210785965330899
	14	87588	1.0e+02*3.89736006335979	0.099506810507876
	16	206172	1.0e+02*3.89680850296493	0.044593973099722

Table 2: Results of numerical solutions of quadratic eigenvalues for region ,with an initial grid of 1/8

	1	2304	56.681054076591394	5.482175956805392
	2	2616	54.063730945910883	2.864852826124881
	4	4512	52.744740755403164	1.545862635617162
	6	8334	52.153313504680995	0.954435384894992
	8	15468	51.745061666227073	0.546183546441071
	10	28248	51.509294035452548	0.310415915666546
	12	53400	51.376378012289926	0.177499892503924
	14	99072	51.290365005390711	0.091486885604709
	15	136656	51.266349425181744	0.067471305395742

Table 3: Results of numerical solutions of quadratic eigenvalues for region ,with an initial grid of 1/8

	1	2304	1.0e+02 *3.614939016628592	32.736159444206237
	2	2838	1.0e+02*3.425149701035349	13.757227884881900
	4	5502	1.0e+02* 3.364259155266177	7.668173307964651
	6	11460	1.0e+02* 3.333056194237199	4.547877205066868
	8	22872	1.0e+02*3.313142058416707	2.556463623017692
	10	45060	1.0e+02* 3.300261319939643	1.268389775311334
	12	89652	1.0e+02*3.294558000135493	0.698057794896329
	14	174900	1.0e+02*3.291024325952683	0.344690376615290
	16	341610	1.0e+02*3.289234693276596	0.165727109006639

Figure 1: On the test domain , the initial grid is 1/8 quadratic adaptive mesh and error curve

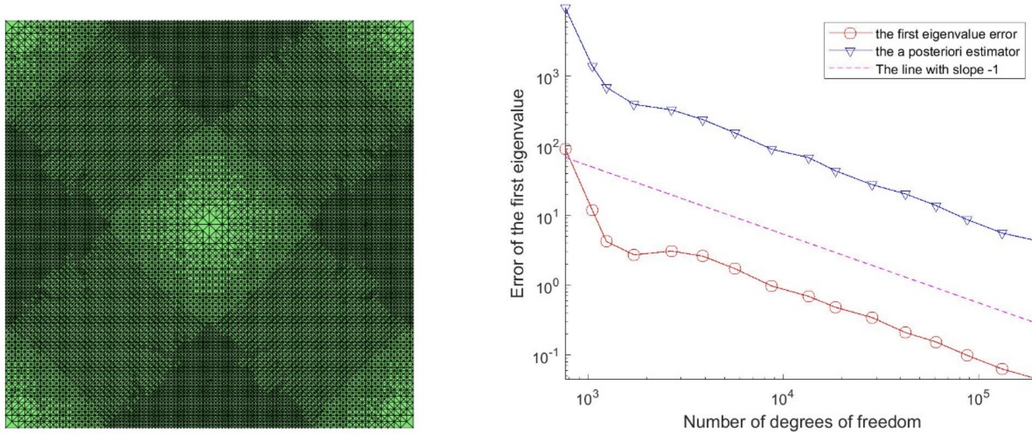


Figure 2: On the test domain , the reference eigenvalue is with an initial grid of 1/8 quadratic adaptive mesh and error curve

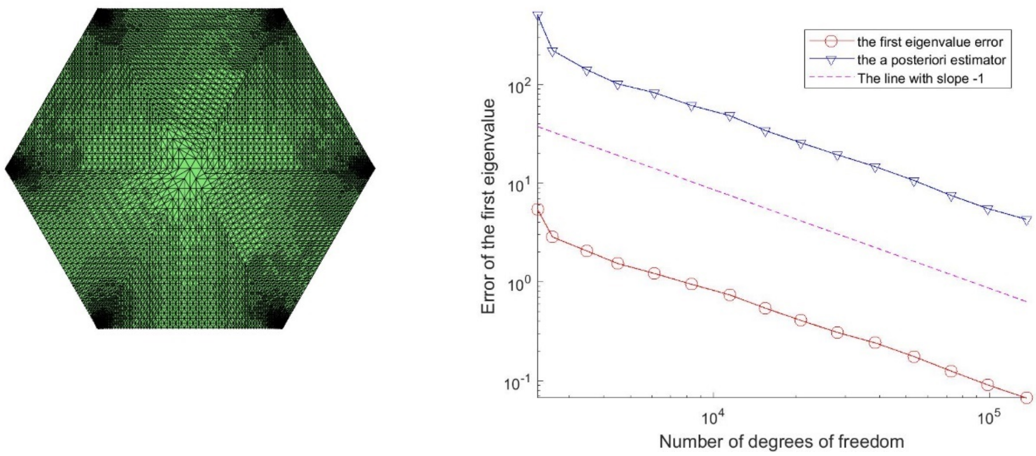
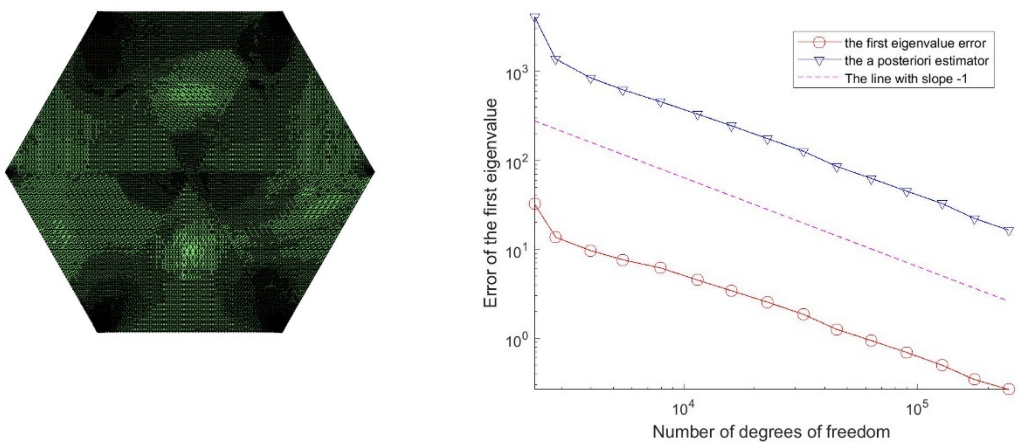


Figure 3: On the test domain , the reference eigenvalue is with an initial grid of 1/8 quadratic adaptive mesh and error curve



The numerical solution results of eigenvalues obtained through adaptive calculation are listed in table 1 to Table 3, and the figure illustrates the adaptive mesh and error curve. From Figure1 to Figure3, we can see that the error curve of the numerical solution for eigenvalues is approximately parallel to the error index curve to a certain extent, the error curve of the quadratic discontinuity element exhibits a nearly parallel relationship with a line having a slope of -1. It shows that all the posterior error indexes of numerical eigenvalues are reliable and effective. The results show that the adaptive algorithm can achieve the optimal convergence order, you can also see from the error curve that for the same degree of freedom, the approximation obtained by the adaptive algorithm is more accurate than that obtained by the uniform grid calculation.

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