Statistical Inference Based on Upper Record Values for the Transmuted Weibull Distribution

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ABSTRACT: In this study, we have considered estimation of unknown parameters based on upper record values for Transmuted Weibull distribution with three parameters. Maximum likelihood and approximate Bayesian estimators based on upper record values for unknown parameters of this distribution are obtained. Tierney-Kadane approximation is used to obtain approximate Bayesian estimators under squared loss function based on upper records. Also, biases and mean square errors of these estimators are compared with a Monte Carlo simulation study. Finally, a data analysis is presented on real data set which fit to transmuted Weibull distribution.

KEYWORDS : Bayesian estimation, maximum likelihood estimation, tierney-kadane approximation, transmuted weibull distribution, upper record values.

Date of Submission: 29-11-2017 Date of acceptance: 12-12-2017

1. INTRODUCTION

The transmuted Weibull (TW) distribution has been introduced by [1]. They have examined statistical properties of this distribution and have showed that this distribution is more flexible according to weibull distribution. Probability density function (pdf) and cumulative distribution function (cdf) of this distribution with \( \mu, \sigma \), and \( \lambda \) parameters are given as follows;

\[
f(x; \mu, \sigma, \lambda) = \frac{\mu}{\sigma} \left( \frac{x}{\sigma} \right)^{\mu-1} \exp \left\{ -\left( \frac{x}{\sigma} \right)^{\lambda} \right\} \left[ 1 - \lambda + 2\lambda \exp \left\{ -\left( \frac{x}{\sigma} \right)^{\lambda} \right\} \right] (1)
\]

\[
F(x; \mu, \sigma, \lambda) = \left[ 1 - \exp \left\{ -\left( \frac{x}{\sigma} \right)^{\lambda} \right\} \right] \left[ 1 + \lambda \exp \left\{ -\left( \frac{x}{\sigma} \right)^{\lambda} \right\} \right] (2)
\]

where \( x > 0, \mu > 0, \sigma > 0 \) and \( \lambda \) is defined on the interval \([-1,1]\).

The first study about record values has been presented by Chandler [2]. In literature, there are many studies based on the record values. Some of these studies are listed as Rényi [3], Arnold et.al. [4], Ahsanullah and Nevzorov [5] and El-Sagheer [6]. The number of studies based on records for transmuted distributions are few [7].

Upper record time and upper record values are defined as follows:

Let \( X_1, X_2, ... \) be a sequence of independent and identically distributed (iid) continuous random variables taken from any distribution having distribution function \( F \). If \( X_i > X_j \) for every \( i < j \), \( X_j \) is called as \( j^{th} \) upper record value. \( n^{th} \) upper record time is defined as follows.

\[
U(n) = \min \left\{ j : i > U(n-1), X_i > X_{U(n-1)} \right\} (3)
\]

where \( U(1) = 1 \) and \( X_{U(1)} \) is \( 1^{st} \) upper record value.

In this paper, our purpose is to compare the performances of maximum likelihood and approximate bayesian estimators of unknown parameters based on upper record values for TW distribution in terms of mean square error (MSE) criteria. This study is organized as follows. In section 2, maximum likelihood estimation (MLE) based on upper record values for TW distribution is given. In section 3, approximate bayesian estimators under square loss function are obtained by using Tierney Kadane approximation. In section 4, Monte Carlo simulation study is performed to compare these estimators in terms of bias and MSE. Moreover, the real data analysis is taken place in section 5. Finally the conclusion are presented in section 6.
II. MLE BASED ON UPPER RECORD VALUES FOR TW DISTRIBUTION

Let \( Y = (X_{(1)}, X_{(2)}, \ldots, X_{(n)}) \) are upper record values taken from TW \((\mu, \sigma, \lambda)\) distribution on condition that \( \lambda \) parameter is known. In this case, likelihood function based on the observed data is as follows.

\[
L(\mu, \sigma \mid Y) = f(Y \mid \mu, \sigma) \prod_{i=1}^{n-1} f(x_{(i+1)} \mid \mu, \sigma) \\
= \left( \frac{\mu}{\sigma} \right)^{n} \prod_{i=1}^{n-1} \left( \frac{x_{(i)}}{\sigma} \right) \exp \left( -\sum_{i=1}^{n} \left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \right) \prod_{i=1}^{n-1} \left[ 1 - \lambda + 2 \lambda \exp \left( -\left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \right) \right]
\]

From (4), log-likelihood function is given as

\[
\ell(\mu, \sigma \mid Y) = n (\log \mu - \log \sigma) + (\mu - 1) \sum_{i=1}^{n} \log \left( \frac{x_{(i)}}{\sigma} \right) - \sum_{i=1}^{n} \left( \frac{x_{(i)}}{\sigma} \right)^{\mu} + \sum_{i=1}^{n} \log \left[ 1 + \lambda + 2 \lambda \exp \left( -\left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \right) \right] - \sum_{i=1}^{n-1} \log \left[ 1 + \lambda + 2 \lambda \exp \left( -\left( \frac{x_{(i+1)}}{\sigma} \right)^{\mu} \right) \right]
\]

Taking the derivative of \( \ell(\mu, \sigma \mid Y) \) with respect to \( \mu \) and \( \sigma \) parameters, following non-linear equations are obtained.

\[
\frac{\ell(\mu, \sigma \mid Y)}{\partial \mu} = \frac{n}{\mu} + \sum_{i=1}^{n} \log \left( \frac{x_{(i)}}{\sigma} \right) - \sum_{i=1}^{n} \left( \frac{x_{(i)}}{\sigma} \right)^{\mu} + \sum_{i=1}^{n} \frac{2 \lambda \left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \log \left( \frac{x_{(i)}}{\sigma} \right) \exp \left( -\left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \right)}{1 - \lambda + 2 \lambda \exp \left( -\left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \right)} + \lambda \left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \log \left( \frac{x_{(i)}}{\sigma} \right) \exp \left( -\left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \right)
\]

\[
\frac{\ell(\mu, \sigma \mid Y)}{\partial \sigma} = -\frac{n}{\sigma} - (\mu - 1) n \frac{\mu}{\sigma} + \sum_{i=1}^{n} \left( \frac{x_{(i)}}{\sigma} \right)^{\mu} + \sum_{i=1}^{n} \frac{2 \lambda \left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \log \left( \frac{x_{(i)}}{\sigma} \right) \exp \left( -\left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \right)}{1 - \lambda + 2 \lambda \exp \left( -\left( \frac{x_{(i)}}{\sigma} \right)^{\mu} \right)} - \sum_{i=1}^{n-1} \mu \left( \frac{x_{(i+1)}}{\sigma} \right)^{\mu} \exp \left( -\left( \frac{x_{(i+1)}}{\sigma} \right)^{\mu} \right)
\]

These equations can be solved by using numerical analysis methods such as newton-raphson method etc. Thus, MLEs for \( \mu \) and \( \sigma \) parameters have been obtained.
III. BAYES ESTIMATION BASED ON UPPER RECORD VALUES FOR TW DISTRIBUTION

Let $X_{10} = (X_{10,1}, X_{10,2}, \ldots, X_{10,n})$ be upper record values taken from TW $(\mu, \sigma, \lambda)$ distribution, where the parameter $\lambda$ is known. Independent Gamma priors for unknown $\mu$ and $\sigma$ parameters needed for bayes estimation are given by:

\[
\begin{align*}
\pi(\mu) &= \mu^{d_1-1} e^{-\mu} d_1, \quad \mu > 0, d_1 > 0, e_1 > 0 \\
\pi(\sigma) &= \sigma^{d_2-1} e^{-\sigma} d_2, \quad \sigma > 0, d_2 > 0, e_2 > 0
\end{align*}
\] 

The joint prior and posterior distributions of $\mu$ and $\sigma$ are given in equation (10) and (11), respectively.

\[
\begin{align*}
\pi(\mu, \sigma) &= \pi(\mu) \pi(\sigma) = \mu^{d_1-1} \sigma^{d_2-1} e^{-\mu - \sigma} d_1 d_2 \\
\pi(\mu, \sigma | \Sigma_0) &= \frac{f(\Sigma_0 | \mu, \sigma)}{f(\Sigma_0)} \pi(\mu) \pi(\sigma)
\end{align*}
\]

where,

\[
\begin{align*}
k(x_{10}; \mu, \sigma) &= \frac{\mu}{\sigma} \\
&= \prod_{i=1}^{n} \left( \frac{x_{10,i}^{\mu}}{\sigma} \right)^{\mu-1} \exp \left( - \sum_{i=1}^{n} \left( \frac{x_{10,i}^{\mu}}{\sigma} \right)^{\mu} \right) \prod_{i=1}^{n} \left[ 1 - \lambda + 2 \lambda \exp \left( - \frac{x_{10,i}^{\mu}}{\sigma} \right) \right] \\
&= \prod_{i=1}^{n} \left[ 1 - \left( 1 - \exp \left( - \frac{x_{10,i}^{\mu}}{\sigma} \right) \right) \right] \prod_{i=1}^{n} \left[ 1 + \lambda \exp \left( - \frac{x_{10,i}^{\mu}}{\sigma} \right) \right]
\end{align*}
\]

In this case, approximate bayes estimator under squared loss function for any function of $\mu$ and $\sigma$, $w(\mu, \sigma)$, is as follows:

\[
\begin{align*}
w_{w}(\mu, \sigma) &= E \left[ w(\mu, \sigma) | \Sigma_0 \right] \\
&= \int_{0}^{\infty} \int_{0}^{\infty} w(\mu, \sigma, \lambda) e^{\left[ \ell(\mu, \sigma, \lambda) - \ell(\mu, \sigma) \right]} d\mu d\sigma \\
&= \frac{1}{n} \left( \rho(\mu, \sigma) + \ell(\mu, \sigma) \right) \\
&= \frac{1}{n} \left( \rho(\mu, \sigma) + \ell(\mu, \sigma) \right)
\end{align*}
\]

where $\rho(\mu, \sigma)$ is defined as follows.

\[
\rho(\mu, \sigma) \equiv \log \pi(\mu, \sigma) = (d_1 - 1) \log \mu + (d_2 - 1) \log \sigma - (\mu e_1 + \sigma e_2)
\]

In this case, approximate Bayes estimator of $w(\mu, \sigma)$ under squared error loss function for TW distribution is derived as follows:

\[
\begin{align*}
\hat{w}_{w}(\mu, \sigma) &= E \left[ w(\mu, \sigma) | \Sigma_0 \right] z^{1/2} \left[ \frac{\det \Sigma}{\det \Sigma} \right] e^{\left[ n \left( \ell(\hat{\mu}, \hat{\sigma}) - \ell(\hat{\mu}, \hat{\sigma}) \right) \right]}
\end{align*}
\]

where $(\hat{\mu}, \hat{\sigma})$ and $(\hat{\mu}, \hat{\sigma})$ maximize $l(\hat{\mu}, \hat{\sigma})$ and $l(\hat{\mu}, \hat{\sigma})$, respectively. $z^*$ and $\Sigma$ are minus the inverse Hessians of $l(\hat{\mu}, \hat{\sigma})$ and $l(\hat{\mu}, \hat{\sigma})$ at $(\hat{\mu}, \hat{\sigma})$ and $(\hat{\mu}, \hat{\sigma})$, respectively.

IV. SIMULATION STUDY
In this section, a Monte Carlo simulation study is carried out to investigate the performances of ML and approximate bayes estimates based on upper record values taken from TW distribution on condition that \( \lambda \) parameter is known. A total of 10000 random samples have been generated from TW distribution with parameters

\[
(\mu = 2, \sigma = 2.4, \lambda = -0.2), (\mu = 3.2, \sigma = 3, \lambda = 0.5), (\mu = 2.1, \sigma = 2.1, \lambda = 0.4), (\mu = 2.7, \sigma = 2.3, \lambda = -0.7).\]

The following algorithm is used to generate upper record values from any \( F \) distribution.

Step 1. \( T_1, T_2, \ldots, T_n \) are generated from \( U(0,1) \) distribution.

Step 2. Data from standard exponential distribution are generated by using \( Z_i = -\ln(1 - T_i) \) transformation.

Step 3. \( i^{th} \) upper record value taken from standard exponential distribution with \( Y_i = Z_1 + Z_2 + \ldots + Z_i \) transformation is generated.

Step 4. \( i^{th} \) upper record value taken from \( U(0,1) \) distribution is obtained by using \( U_i = 1 - \exp(-T_i) \) transformation.

Step 5. Finally, upper record values taken from any distribution \( F \) are generated with \( X_{(U_i)} = F^{-1}(U_i) \) transformation.

Mean square error (MSE) and biases of the ML and approximate bayes estimates based on upper record values using \((5)\) and \((16)\) equations are shown in Table 1.

**Table 1.** MSEs and biases of the ML and approximate bayes estimates for various parameter values and \((d_i = 1, d_j = 2, e_i = 1, e_j = 2)\) priors

<table>
<thead>
<tr>
<th>n</th>
<th>Parameter values</th>
<th>( \mu )</th>
<th>MSE</th>
<th>Bias</th>
<th>MSE</th>
<th>Bias</th>
<th>( \mu )</th>
<th>MSE</th>
<th>Bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>(2.2,4,-0.2)</td>
<td>7.220</td>
<td>1.347</td>
<td>1.400</td>
<td>0.499</td>
<td>0.351</td>
<td>-0.438</td>
<td>0.652</td>
<td>-0.727</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>1.118</td>
<td>0.493</td>
<td>1.180</td>
<td>0.410</td>
<td>0.229</td>
<td>-0.352</td>
<td>0.595</td>
<td>-0.590</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>0.545</td>
<td>0.307</td>
<td>1.025</td>
<td>0.348</td>
<td>0.173</td>
<td>-0.297</td>
<td>0.544</td>
<td>-0.540</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0.329</td>
<td>0.223</td>
<td>0.865</td>
<td>0.305</td>
<td>0.136</td>
<td>-0.259</td>
<td>0.490</td>
<td>-0.509</td>
</tr>
<tr>
<td>25</td>
<td></td>
<td>0.238</td>
<td>0.169</td>
<td>0.777</td>
<td>0.259</td>
<td>0.116</td>
<td>-0.235</td>
<td>0.465</td>
<td>-0.493</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>0.182</td>
<td>0.140</td>
<td>0.649</td>
<td>0.196</td>
<td>0.097</td>
<td>-0.208</td>
<td>0.443</td>
<td>-1.026</td>
</tr>
<tr>
<td>5</td>
<td>(3.2,3,0.5)</td>
<td>17.732</td>
<td>2.125</td>
<td>0.629</td>
<td>0.241</td>
<td>1.576</td>
<td>-1.165</td>
<td>1.184</td>
<td>-0.925</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>3.009</td>
<td>0.822</td>
<td>0.610</td>
<td>0.238</td>
<td>1.008</td>
<td>-0.893</td>
<td>0.999</td>
<td>-0.819</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>1.509</td>
<td>0.523</td>
<td>0.564</td>
<td>0.231</td>
<td>0.749</td>
<td>-0.735</td>
<td>0.847</td>
<td>-0.732</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0.916</td>
<td>0.382</td>
<td>0.494</td>
<td>0.206</td>
<td>0.581</td>
<td>-0.626</td>
<td>0.719</td>
<td>-0.672</td>
</tr>
<tr>
<td>25</td>
<td></td>
<td>0.663</td>
<td>0.290</td>
<td>0.452</td>
<td>0.172</td>
<td>0.484</td>
<td>-0.554</td>
<td>0.644</td>
<td>-0.637</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>0.501</td>
<td>0.233</td>
<td>0.385</td>
<td>0.121</td>
<td>0.403</td>
<td>-0.492</td>
<td>0.590</td>
<td>-0.171</td>
</tr>
<tr>
<td>5</td>
<td>(2.1,2,1.04)</td>
<td>7.863</td>
<td>1.141</td>
<td>0.861</td>
<td>0.341</td>
<td>0.334</td>
<td>-0.343</td>
<td>0.369</td>
<td>-0.500</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>1.271</td>
<td>0.536</td>
<td>0.808</td>
<td>0.336</td>
<td>0.226</td>
<td>-0.296</td>
<td>0.361</td>
<td>-0.464</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>0.599</td>
<td>0.336</td>
<td>0.704</td>
<td>0.297</td>
<td>0.170</td>
<td>-0.256</td>
<td>0.336</td>
<td>-0.423</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0.385</td>
<td>0.245</td>
<td>0.624</td>
<td>0.259</td>
<td>0.139</td>
<td>-0.226</td>
<td>0.318</td>
<td>-0.390</td>
</tr>
<tr>
<td>25</td>
<td></td>
<td>0.276</td>
<td>0.191</td>
<td>0.549</td>
<td>0.224</td>
<td>0.118</td>
<td>-0.203</td>
<td>0.300</td>
<td>-0.365</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>0.219</td>
<td>0.150</td>
<td>0.467</td>
<td>0.151</td>
<td>0.105</td>
<td>-0.183</td>
<td>0.301</td>
<td>-0.358</td>
</tr>
<tr>
<td>5</td>
<td>(2.7,2.3,0.7)</td>
<td>11.453</td>
<td>1.682</td>
<td>0.587</td>
<td>0.307</td>
<td>0.714</td>
<td>-0.710</td>
<td>0.509</td>
<td>-0.642</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>1.832</td>
<td>0.629</td>
<td>0.485</td>
<td>0.245</td>
<td>0.460</td>
<td>-0.507</td>
<td>0.421</td>
<td>-0.542</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>0.900</td>
<td>0.395</td>
<td>0.424</td>
<td>0.207</td>
<td>0.344</td>
<td>-0.399</td>
<td>0.361</td>
<td>-0.463</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0.536</td>
<td>0.285</td>
<td>0.362</td>
<td>0.181</td>
<td>0.264</td>
<td>-0.336</td>
<td>0.312</td>
<td>-0.410</td>
</tr>
<tr>
<td>25</td>
<td></td>
<td>0.403</td>
<td>0.225</td>
<td>0.329</td>
<td>0.155</td>
<td>0.223</td>
<td>-0.291</td>
<td>0.287</td>
<td>-0.373</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>0.307</td>
<td>0.191</td>
<td>0.282</td>
<td>0.123</td>
<td>0.186</td>
<td>-0.252</td>
<td>0.263</td>
<td>-0.353</td>
</tr>
</tbody>
</table>

V. REAL DATA ANALYSIS

In this section, we provide a real data analysis in order to indicate fit to the TW \((\mu, \sigma, \lambda)\) distribution.

We consider real data set based on the breaking stress of carbon fibers studied by Nichols and Padgett, [12], Pal et. al., [13], Aryal and Tsokos, [3]. The data set consists of 100 observations and given as follows.

Data Set

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

Firstly, we have examined whether this data set fit to the TW distribution. The parameters of this distribution is estimated by maximum likelihood method. The estimated parameters and their standard errors (in parentheses),

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Kolmogorov-Smirnov (K-S) distances between the theoretic and empirical distribution functions, and p-values are given in Table 2.

<table>
<thead>
<tr>
<th>$\hat{\mu}$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\lambda}$</th>
<th>K-S</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.9935 (0.2413)</td>
<td>3.4125 (0.3377)</td>
<td>0.6789 (0.3798)</td>
<td>0.0642</td>
<td>0.8038</td>
</tr>
</tbody>
</table>

From Table 2, it is clear that the data set fits to the TW distribution. Also, figure 1 shows plots of the empirical and theoretic distribution functions.

From the data set, the observed upper record values is obtained as follows.

$x_{U(i)} : 3.70, 4.42, 4.90, 4.91, 5.56$.

The ML estimates and goodness of fit measures of TW distribution for the real data set are given in Table 2.

Table 2. ML estimates and goodness of fit measures of TW distribution for the real data set

![Empirical CDF and theoretic CDF](image)

From the data set, the observed upper record values is obtained as follows.

$x_{U(i)} : 3.70, 4.42, 4.90, 4.91, 5.56$.

The ML estimates and standard errors for unknown parameters of TW $(\mu, \sigma, \lambda)$ distribution based on upper record values are given in Table 3.

Table 3. ML (standard error) for $(\mu, \sigma, \lambda)$ parameters based on upper record values

<table>
<thead>
<tr>
<th>$\hat{\mu}$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.5017 (2.7191)</td>
<td>4.5157 (0.5464)</td>
<td>0.727 (0.6379)</td>
</tr>
</tbody>
</table>

VI. CONCLUSION

In this paper, we have studied about estimation problem of parameters based on upper record values for TW distribution on condition that transmuted parameter is known. MLEs of unknown parameters are obtained. Also approximate bayesian estimates under squared loss function of these parameters are derived using Tierney Kadane method. It has been carried out a Monte-Carlo simulation study to compare these estimators in terms of mse and bias. According to simulation results, it is clear that performances of the approximate bayes estimators are better than maximum likelihood estimator in terms of mse and bias. Also, as the number of record values increases, the values of these two estimators approach to each other. In the real data analysis, ML estimate based on upper record values generated from Carbon fibres data set is calculated.

REFERENCES
