International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 5 Issue 8 || October. 2017 || PP-30-36

Fibonacci Matrix Summability of Fourier series

*****Ahmadu Kiltho, Abdullahi Mohammed and Ado Balili
**Department of Mathematical Sciences, University of Maiduguri, Borno State, Nigeria*

ABSTRACT: *This note aims at applying recently defined infinite matrix method which we apply in the summation of Fourier series. The summation amounts to uniform convergence of Fourier series as maintained by Fejer, since1904.*
Keywords: *Fibonacci matrix, Fourier series, Matrix method, Uniform convergence,*
Mathematical Subject Classification*: 40A05,40C05, 40D05*

Date of Submission: 23-11-2017 Date of acceptance: 06-12-2017

I. INTRODUCTION

In 2003, Kalman and Mena wrote that among numerical sequences the Fibonacci numbers achieved a kind of celebrity status, because they are famous for possessing wonderful and amazing properties – some are well known. Among these properties, are that the difference of two Fibonacci numbers is a Fibonacci number, ratios of Fibonacci numbers converge to the golden mean, any four consecutive Fibonacci numbers are Fibonacci numbers, the greatest common divisor, *gcd*, of two Fibonacci numbers is another Fibonacci number, just to mention but few. So, Fibonacci numbers stand out to be a kind of super sequence. To start with, Fibonacci sequence f_n say, are the terms of the sequence $0,1,1,2,3,5,\ldots$ wherein each term is the sum of the two preceding terms starting with 0 and 1 denoted by f_0 and f_1, respectively. The name Fibonacci sequences is due to Francois Edwouard Anatole Lucas in 1876. Sum of squares, asymptotic behavior, running sums, finite matrix form of Fibonacci numbers can be seen in [15]. Recently, infinite matrices generated by Fibonacci numbers has been used in the works of [16].

The book by [18], version 2002 I feel, is the principal authority in the studies of Fourier series, wherein he also wrote a certain form of trigonometrical series called Fourier series of form:

$$\frac{1}{2}a_0 + \sum_{v=1}^{\infty}(a_v \cos vx + b_v \sin vx) \qquad (1)$$

where the coefficients $a_0, a_1, \ldots, b_1, b_2, \ldots$ are independent of *x*. The coefficients are real, and since all the terms of (1) are of period 2π, it is sufficient to study trigonometrical series in any interval of length 2π, for example, $(0,2\pi)$ or $(-\pi,\pi)$. The system of functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$, called the trigonometrical system is orthogonal in $(-\pi,\pi)$. Furthermore, let

$I_{m,n} = \int_{-\pi}^{\pi} \sin mx \sin nx$ and $I'_{m,n}$, $I''_{m,n}$ denoting the corresponding integrals with $\cos mx \sin nx$ and $\cos mx \cos nx$. Integrating the formula

$2\sin mx \sin nx = \cos(m-n)x - \cos(m+n)x$ and taking into account the periodicity of integral functions, then we find that $I_{m,n} = 0$ whenever $m \neq n$. This is true even when $m = n$. The λ's are now $2\pi, \pi, \pi, \ldots$, and so, if for a given function f we put

$$a_v = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos vt\,dt, \qquad b_v = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin t\,dt \qquad (2)$$

Then (1) is called the Fourier series of f. On changing the definition of Fourier series in the case of a_0, we shall call a_v and b_v the Fourier coefficients of the function f. So many works on the study of Fourier series and their convergence were carried out by several authors starting from [5] who worked mainly on Fourier series. This paper cannot contain all the references to research works in this respect starting from late 1800's. However, the following periodical papers as grouped below were enough for this piece of work:

On summability and convergence of Fourier series see [2], [3], [4], [6] and [17]; on Cesaro summability of Fourier series see [6], [7], [8] and [9]; on linear and triangular methods of summability see [11] and [12]; and on matrix methods of summability see [13] and [14].
In 1911, Teoplitz, O. published some work concerning infinite matrices with conditions. They were matrices of the form:

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \ldots a_{0n} & \ldots \\ a_{10} & a_{11} & a_{12} \ldots a_{1n} & \ldots \\ \ldots & \ldots & \ldots \ldots\ldots & \ldots \\ a_{n0} & a_{n1} & a_{n2} \ldots a_{nn} & \ldots \\ \ldots & \ldots & \ldots \ldots\ldots & \ldots \end{bmatrix}$$

The matrix A is called a Teoplitz matrix or T-matrix if

(i) $\lim_{i \to \infty} a_{iv} = 0, \ v = 0, 1, 2, \ldots,$

(ii) $\lim_{i \to \infty} A_i = 1$ and

(iii) $N_i \le C, \ i = 0, 1, 2, \ldots$

where C is independent of i, see ([18], p. 39).
Summability is about generalization of the convergence of sequences and series. The use of infinite matrices to realize convergence cannot be over emphasized. Any infinite matrix used in summability of sequences and series is called a method of summability, for instance the (C, 1) method called the Cesaro method of summability, where the limit of a sequence (x_n) according to Cesaro can be defined as $\lim y_n$, where (y_n) is given by $y_n = \frac{1}{n+1}\sum_{i=0}^{n} x_i$. In [5] Fejer's theorem on the other hand states that if a function $f \in L^p(0, 2\pi), 1 \le p < \infty$ then the Cesaro's means y_n of the partial sum of the Fourier series of f converges to f in the L^p - norm. If in addition ,f is continuous and $f(\pi) = f(-\pi)$, then y_n converges uniformly to f . Several generalizations of (C, 1) method led to the methods (C, k), up to (C, α). The work of Bosanquet, see [1] is regarded as the best generalization of (C, α) method. We remark that it is natural replace to replace (C, 1) method with any other methods. We wish to replace with Fibonacci infinite matrix denoted by F, defined in [16] as follows:

$$F = (f_{nv})_{n,v=1}^{\infty} = \begin{cases} \frac{f_v^2}{f_n f_{n+1}}, & \text{for } 1 \le v \le n \\ 0, & \text{for } v > n \end{cases} \qquad (3)$$

It is a triangle, that is, $f_{nn} \ne 0$ and $f_{nv} = 0$ for $v > n \ (n = 1, 2, 3, \ldots)$. It is also a regular matrix, for it satisfies the condition of regular matrices as spelt out in [19], that any matrix $A = (a_{nv})_{n,v=1}^{\infty}$ is regular if and only if

(i) there exists $M > 0$, such that for every $n = 1, 2, 3, \ldots \sum_{v=1}^{\infty} |a_{nv}| \le 1$

(ii) $\lim_{n \to \infty} a_{nv} = 0$ for every $v = 1, 2, \ldots$ and

(iii) $\lim_{n \to \infty} \sum_{v=1}^{\infty} a_{nv} = 1.$

Clearly, the Fibonacci matrix F is regular. We wish to establish an analogue that is replaced by (C, 1) method is replaced F, then F –transform of the of the partial sum of the Fourier series of 2π -period continuous function f converges uniformly to f. To do this, we need to fix notations and some necessary preceding lemmas to the main result of this paper.

II. BASIC DEFINITIONS AND LEMMAS

Definition 2.1 $C_{2\pi} = \{ f : \mathbb{R} \to \mathbb{R} \mid f \ is\ 2\pi - \text{periodic and continuous}\}$, the set of all continuous, 2π -periodic functions. If any f is 2π -periodic and $f \in (-\pi, \pi)$ is Riemann integrable, then (2) holds. If $v = 0$, then $b_v = 0$ and the Fourier series (1) is the Fourier series of f with Fourier coefficients a_v and b_v .For each $f \in C_{2\pi}$ and $n \in \mathbb{N}$ the equality

$$s_n(f,t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) D_v(t-x)dx$$

where , $D_v(t) = \dfrac{\sin(n+\frac{1}{2})t}{2\sin\frac{v}{2}}$, $t \in \mathbb{R}$, $\frac{v}{2} \neq v\pi$ is called the Dritchlet kernel. And $s_n(f,t)$ the *n*th partial sum of the Fourier series (1). D_v has the following properties:

Lemma 2.1: $(i)\ D_v \in C_{2\pi}$ for all $v \in \mathbb{N}$, and $(ii)\ |D_v(t)| \leq v + \frac{1}{2} < v+1$ for any $v \in \mathbb{N}$ and $t \in \mathbb{R}$.

In the equation:

$$\sigma_n^F(f,t) = \sum_v f_{nv} s_v(f,t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) K_n^F(t-x)dx, \text{ with } K_n^F(t) = \sum_v a_{nv} D_v(t) \tag{4}$$

$\sigma_n^F(f,t)$ is the sequence of partial sums of the Fourier series of any 2π – periodic continuous function *f*, while $K_n^F(t)$ is the *n*th kernel corresponding to the Fibonacci matrix, *F*. We wish to show that $\sigma_n^F(f,t)$, the transform of $s_n(f,t)$ under *F* converges uniformly to *f*.

We wish to adopt the following notations:

$$\sigma_n^F(x; f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t) K_n^F(t)dt \tag{5}$$

$$\int_{-\pi}^{0} f(x+t)K_n^F(t)dt = \int_0^{\pi} f(x-t) K_n^F(t)dt \tag{6}$$

Note that $K_n^F(-t) = K_n^F(t)$, for the kernel $K_n^F(t)$ is 2π – periodic and continuous function. Also,

$$\sigma_n^F(x; f) = \frac{2}{\pi}\int_0^{\pi} \left\{ \frac{f(x+t) + f(x-t)}{2} \right\} K_n^F(t)dt \tag{7}$$

Let

$$\sigma_f(x) = \lim_{t \to 0^+} \frac{f(x+t) + f(x-t)}{2} \tag{8}$$

This function is defined at those points for which the preceding limit exists. If *f* is continuous at some point *x* then $\sigma_f(x) = f(x)$; and if it has a jump discontinuity at x_0 , then

$$\sigma_f(x_0) = \frac{f(x_0+) + f(x_0-)}{2} \quad .$$

Lemma 2.2 (Kalman and Mena, [15]): Let (f_n) be a sequence of Fibonacci numbers. Then their sum of squares

$$\sum_{v=1}^{n} f_v^2 = f_n f_{n+1} \text{ and } \frac{f_n + 1}{f_n} \to 0 \text{ as } n \to \infty$$

Lemma 2.3 (Kara and Basarir, [16]): Let (f_n) be a sequence of Fibonacci numbers. Then

$(i)\ \sum_{v=1}^{\infty} \frac{1}{f_v}$, converges, and $(ii)\ \sup_v \left(f_v^2 \sum_{n=v}^{\infty} \frac{1}{f_n f_{n+1}} \right) < \infty$

Lemma 2.4: For any bounded sequence such that $|x_v| \le M$, $v \in$, we have

$$\left| \frac{1}{f_n f_{n+1}} \sum_{v=1}^{n} f_v^2 x_v \right| \le \frac{1}{f_n f_{n+1}} \sum_{v=1}^{n} f_v^2 |x_v|$$

$$\le \frac{M}{f_n f_{n+1}} \sum_{v=1}^{n} f_v^2 = M .$$

Lemma 2.5 ([17] p.44): Let *n*th kernel of an identity infinite matrix be denoted by $K_n(t)$ satisfying the following conditions:

$(a)\ \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1$,

$(b)\ K_n(t) \ge 0$,

$(c)\ \frac{1}{\pi}\int_{-\pi}^{\pi} |K_n(t)|\, dt \le C$ with *C* independent of *n*, and

(d) if $\mu_n(\delta) = \max_{\delta \le t \le \pi} |K_n(t)|$, $(0 < \delta \le \pi)$ then $\mu_n(\delta) \to 0$ for each fixed δ. Then the *n*th kernel is positive if it satisfies conditions (*a*) and (*b*); and is called quasi-positive if it satisfies condition (*c*) only.

Lemma 2.6 ([18], p.46): (i) If $K_n(t)$ is a positive kernel, then for any *f* satisfying $m \le f \le M$, we have $m \le \sigma_n(x, f) \le M$. (ii) If $K_n(t)$ is a quasi-positive kernel, and $|f| \le M$ then it implies that $|\sigma_n(x, f)| \le CM$, with *C* as in condition (*c*).

The last Lemma and equation (4) above suggest that we must perform the calculations for $K_n^F(t)$, the *n*th kernel corresponding to the Fibonacci matrix, *F* defined in (3),and subsequently for $\sup_n L_n^F$ and $\mu_n(\delta) = \max_{\delta \le t \le \pi} |K_n^F(t)|$. So, we should have:

$$K_n^F(t) = \sum_{v=0}^{n} a_{nv} D_v$$

$$= \sum_{v=0}^{n} \frac{f_v^2}{f_n f_{n+1}} D_v$$

$$= \frac{1}{f_n f_{n+1}} \sum_{v=1}^{n} f_v^2 D_v$$

where,

$$D_v(t) = \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{v}{2}}$$

$$\Rightarrow \qquad K_n^F(t) = \frac{1}{f_n f_{n+1}} \sum_{v=1}^{n} f_v^2 \cdot \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{v}{2}}, \quad n \ge 1$$

$$= \frac{1}{f_n f_{n+1}} \sum_{v=1}^{n} f_v^2 \cdot \sum_{v=1}^{n} \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{v}{2}}, \quad n \ge 1$$

$$= \frac{1}{2}\cdot\sum_{v=1}^{n}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{v}{2}}$$

$$= \frac{1}{2\sin\frac{v}{2}}\cdot\sum_{v=0}^{n}\sin\frac{t}{2}\sin(v+\frac{1}{2})$$

$$= \frac{2}{2\sin\frac{v}{2}}\cdot\sum_{v=0}^{n}(\cos vt-\cos(v+1))t$$

$$= \frac{1}{\sin\frac{v}{2}}\cdot(v1-\cos(v+1)t$$

or, $$K_n^F(t) = \left(\frac{\sin(\frac{v+1}{2})t}{\sin\frac{t}{2}}\right)^2 \geq 0 \qquad (9)$$

$\Rightarrow$ $$|K_n^F(t)| = \left|\left(\frac{\sin(\frac{v+1}{2})t}{\sin\frac{t}{2}}\right)\right|^2$$

$$\leq \left|\frac{1}{\sin\frac{t}{2}}\right|^2$$

$$\leq \left|\frac{1}{\sin^2\frac{t}{2}}\right| \qquad (10)$$

$$\leq 1$$

Also,

$$\mu_n(\delta) = \max_{\delta\leq t\leq\pi}|K_n^F(t)|, \qquad 0<\delta\leq\pi$$

$$\leq \frac{1}{n}\cdot\frac{1}{\sin\frac{\delta}{2}} \to 0, \quad \text{for } n\geq 1$$

III. MAIN RESULT

Theorem 3.1:*Let* $f \in C[-\pi,\pi]$ *be such that* $s(f,x) = s_n(f,x)$ for $n \in \square$, $x \in \square$ *and a matrix* $F = (f_{nv})$ *satisfying*

$(i)\sum_{v}(v+1)|f_{nv}| < \infty$,

(ii) $\sum_{v} f_{nv} = 1$,

(iii) $\sup_{n} L_n^F = \frac{2}{\pi}\int_0^{\pi}|K_n^F(t)|\,dt < \infty$ *, and*

(iv) $\mu_n(\delta) = \max_{\delta \le t \le \pi} |K_n^F(t)| \to 0,$ *for all* $\delta \in (0,1)$. *Then* $\forall\ f \in C[-\pi,\pi],\ t \in$, *we have:* $\sigma_n^F(f,x) \to \sigma_f(x)$, *uniformly.*

Proof: Assume that $\sigma_f(x)$ exists at some point $x \in [0, 2\pi]$. Then there exists some $0 < \delta \le \pi$ such that $\left| \frac{f(x+t)+f(x-t)}{2} - \sigma_f(x) \right| < \varepsilon$, for all $0 < t < \delta$. In view of uniform continuity of *f*, we can select a δ to be independent of $x \in [a,b]$. So, from (5) and (7) we have,

$$\left| \sigma_n^F - \sigma_f(x) \right| = \frac{2}{\pi} \left| \int_0^\pi \left[\frac{f(x+t)+f(x-t)}{2} - \sigma_f(x) \right] K_n^F(t)dt \right|$$

$$\le \frac{2}{\pi} \int_0^\delta \left| \frac{f(x+t)+f(x-t)}{2} - \sigma_f(x) \right| K_n^F(t)dt$$

$$+ \frac{2}{\pi} \int_\delta^\pi \left| \frac{f(x+t)+f(x-t)}{2} - \sigma_f(x) \right| K_n^F(t)dt$$

Let us perform some necessary estimates as follows:

$$\frac{2}{\pi} \int_0^\delta \left| \frac{f(x+t)+f(x-t)}{2} - \sigma_f(x) \right| K_n^F(t)dt$$

$$\le \frac{2}{\pi} \int_0^\delta \varepsilon K_n^F(t)dt$$

$$\le \frac{2}{\pi} \int_0^\pi \varepsilon K_n^F(t)dt \le \varepsilon \qquad (11)$$

And, since

$$K_n^F(t) = \frac{1}{4n} \left(\frac{\sin(\frac{n+1}{2})t}{\sin \frac{t}{2}} \right)^2$$

$$\le \frac{1}{4n \sin^2 \frac{\delta}{2}}, \text{ for each } \delta \le t \le \pi.$$

This implies that

$$\left| \sigma_n^F - \sigma_f(x) \right| \le \frac{1}{4n \sin^2 \frac{\delta}{2}} \int_0^\pi \left| \frac{f(x+t)+f(x-t)}{2} - \sigma_f(x) \right| dt \le \frac{H}{4n} \qquad (12)$$

where,

$$H = \frac{1}{\pi \sin^2 \frac{\delta}{2}} \int_0^\pi \left| \frac{f(x+t)+f(x-t)}{2} - \sigma_f(x) \right| dt$$

Clearly, *H* is independent of *x*. So, for some *N*, choose $\frac{H}{4n} < \varepsilon$, then the estimates in (11) and (12) yield $\left| \sigma_n^F - \sigma_f(x) \right| < \varepsilon + \varepsilon \le 2\varepsilon$, is true for all $n \ge N$ and for all $x \in [a,b]$ provided f is continuous on $[-\pi,\pi]$. Thus $\sigma_n^F \to \sigma_f(x)$, uniformly.

Q.E.D.

IV. CONCLUSION

In conclusion we have successively used Fibonacci Matrix of recent origin to realize uniform convergence of Fourier series, a method different from Fejér since 1904. Thus the contribution is new method, or proof, to show uniform convergence of Fourier series.

ACKNOWLEDGEMENT

We would like to express our gratitude to the reviewer for his/her careful reading and valuable suggestions which improved the representation of the paper.

REFERENCES

[1]. L. S. Bosanquet, On the study of Fourier series, *Proceedings of London Mathematical Society, (31)* 1930, 144 – 164
[2]. E. Hille, and J. D. Tamarkin, On the study of Fourier series, *Transactions of American Mathematical Society, (34)* 1932, 757 – 783
[3]. O Szasz, Convergence properties of Fourier series, *Transactions of American Mathematical Society,* 1934, 483 – 500
[4]. L. S. Bosanquet, Note on Fourier series, *Composito Mathematica, (1)* 1935, 180 – 187
[5]. S. Izumi, Some trigonometrical series VI, *Tohoku Journal of Mathematics, 5 (3)* 1954,290 – 295
[6]. L. Fejér, Untersuchungen über Fouriesche, *Mathematische Annalen,* 1904, 51 – 69
[7]. S. Yano, Note on Fourier analysis XXXI: Cesaro summability of Fourier series, *Pacific Journal of Mathematics,(2)* 1952, 419 – 429
[8]. S. Yano, Cesaro summability of Fourier series, *Tohoku Journal of Mathematics, 5 (3)* 1953, 196 – 197
[9]. K. Kanno On the Cesaro summability of Fourier series (II), *Tohoku Journal of Mathematics, (7)* 1955, 110 – 118
[10]. K. Yano, On a method of Cesaro summation of Fourier series, *Kodai Math. Sem.Rep.. 9 (2)* 1956, 49 – 58
[11]. A. V Effimov, On linear summability methods for Fourier series, *Akad. NaukSSSR Ser. Mat.. 24, (5)* 1960, 743 – 756
[12]. H. P. Dikshit,, Summability of Fourier series by triangular matrix transformations, *Pacific Journal of Mathematics, 30, (2)* 1969, 399 – 410
[13]. A. M.Jarrah, and E. Malkowsky, Ordinary, absolute and strong summability and matrix transformations, *Filomat, (17)* 2003, 59 – 78
[14]. G. H. Hardy, *Divergent series,* Oxford University Press, London, 1949
[15]. D. Kalman, and R. Mena, The Fibonacci numbers-Exposed, *Mathematics Magazine,76, (3)* 2003, 167 – 181
[16]. E. E. Kara, and M. Basarir, An application of Fibonacci numbers into infinite Toeplitz matrices, *Caspian Journal of Mathematical Sciences, 1 (1)* 2012, 43 – 47
[17]. A. Zygmund, A remark on conjugate series, *Proceedings of London Mathematical Society, (31)* 1932, 435 – 436
[18]. A. Zygmund, *Trigonometrical series,* Dover Publications, USA, 1955
[19]. A. Wilansky, *Summability through Functional Analysis*, Math. Studies, North - Holland, 1984

Ahmadu Kiltho "Fibonacci Matrix Summability of Fourier series." International Journal of Mathematics and Statistics vention(IJMSI), vol. 5, no. 8, 2017, pp. 30-36.