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New Generalized Difference Sequence Spaces Generated By Using Riesz Mean and Musielak-Orlicz function

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ABSTRACT: *In this paper we introduce new generalized difference sequence spaces $r^q(\mathcal{M}, \Delta_r^m, u, p, s)$ by using Riesz mean and Musielak-Orlicz function. We also study some topological properties, inclusion relations and finally compute the $\alpha-, \beta-$ and $\gamma-$ duals of these spaces.*

KEYWORDS: *Orlicz function, Paranormed Sequence Spaces, Riesz Sequence space, Sequence Space of Non Absolute Type,*
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I. INTRODUCTION

We denote the set of all sequences (real or complex) by ω. Any subset of ω is called the sequence space. So the sequence space is the set of scalar sequences (real or complex) which is closed under coordinate wise addition and scalar multiplication.

Throughout the paper $\mathbb{N}, \mathbb{R}, and\ \mathbb{C}$ denote the space of all non negative integers, the space of all real numbers and the space of all complex numbers respectively. Let $\ell_\infty, c\ and\ c_0$ respectively denotes the space of all bounded sequences, the space of all convergent sequences and the space of all sequences converging to zero. Also by $\ell_1, \ell(p), cs\ and\ bs$ we denote the space of all absolutely, p- absolutely convergent, convergent, and bounded series respectively. We use the convention that any term with negative subscript equal to zero.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) \in \omega : Ax \in X\} \tag{1.1}$$

Let (q_k) be a sequence of positive numbers and let us write $Q_k = \sum_{k=0}^{n} q_k$ for all positive integers. Then the matrix $R^q = (r_{nk}^q)$ of Riesz mean (R, q_n) is given by

$$r_{nk}^q = \begin{cases} \dfrac{q_k}{Q_k} & if\ 0 \le k \le n \\ 0 & if\ k > n \end{cases}$$

The Riesz mean (R, q_n) is regular if $Q_n \to \infty, as\ n \to \infty$ (see Petersen [2,p.10].

The sequence space $r^q(u, p)$ introduced by Sheikh and Ganie [3] as

$$r^q = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{p_k} < \infty \right\}$$

Where $0 \le p_k \le D < \infty$.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $\sup_k p_k = D,\ and\ H = \max\{1, D\}$. Then, the linear spaces $\ell(p)\ and\ \ell_\infty(p)$ were defined by Maddox [4], as follows:

$$\ell(p) = \{x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty\}$$

$$\ell_\infty(p) = \{x = (x_k) \in \omega : \sup_k |x_k|^{p_k} < \infty\}$$

Which are complete spaces paranormed by

$$g_1(x) = [\sum_k |x_k|^{p_k}]^{1/H}\ and\ \ g_2(x) = \sup_k |x_k|^{p_k/H}$$

If and only if $inf p_k > 0. for\ all\ k$.

Throughout the paper we shall assume that $p_k^{-1} + \{p_k^1\}^{-1} = 1$ provided $1 < inf p_k \le D < \infty$ and we denote the collection of all finite subset of $\mathbb{N}\ by\ \mathcal{F}\ where\ \mathbb{N} = \{0,1,2, \ldots\}$.

Orlicz function is defined as the function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [7] used the concept of Orlicz functions to define the space

$$\ell_M = \left\{x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty\right\}. \tag{1.2}$$

called Orlicz sequence space, and proved that every Orlicz sequence space contains a subspace isomorphic to $\ell_p (1 \le p < \infty)$. The sequence space ℓ_M defined in (1.2) is a Banach space with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\} \tag{1.3}$$

It is shown in [8] that every Orlicz sequence space l_M contains a subspace isomorphic to $\ell_p (p \ge 1)$

An Orlicz function is said to satisfy the $\Delta_2 - condition$ for all values of u if there exists a constant k > 0 such that $M(2u) \le kM(u), u \ge 0$. The $\Delta_2 - condtion$ is equivalent to $M(nu) \le knM(u)$, for all values of u and $n > 1$.

A sequence space $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak –Orlicz function. (see [8],[9]).

A sequence $\aleph = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|.u - M_k(u) : u \ge 0\}\, k = 1,2, \ldots$$

Is called the complimentary function of a Musielak-Orlicz function M. for a given Musielak-Orlicz function M, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follow

$$t_{\mathcal{M}} = \{x \in \omega : I_M(cx) < \infty, \infty for\ all\ c > 0\}$$

$$h_{\mathcal{M}} = \{x \in \omega : I_M(cx) < \infty, for\ all\ c > 0\}$$

Where I_M is a convex modular defined by

$$I_M = \sum_{k=1}^{\infty} M_k(x_k)\ and\ x = (x_k) \in t_{\mathcal{M}}$$

Consider $t_{\mathcal{M}}$ equipped with Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_M\left(\frac{x}{k}\right) \le 1\right\}$$

Or equipped with the Orlicz norm

$$\|x\| = \inf\left\{\frac{1}{k}\left(1 + I_M(kx)\right) : k > 0\right\}.$$

The notion of difference sequence spaces was introduced by Kizmaz [10], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. This notion was further generalized by Et and Colak [11] defined the sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Et and Esi [4], then defined the following spaces:

$$Z(\Delta_n^m) = \{x = (x_k) \in \omega : (\Delta_n^m x_k) \in Z\}$$

For $Z = c, c_0, and\ l_\infty$ where

$\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$, and $\Delta_n^0 x_k = x_k$

For all $k \in N$, which is equivalent to binomial representation

$$\Delta_n^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{+ni}$$

It was proved that the generalized sequence space $Z(\Delta_n^m)$, where $Z = \ell_\infty, c$ or c_0, is a Banach space with norm defined by

$\|x\|_{\Delta_n^m} = \sum_{i=1}^{m} |x_i| + sup|\Delta_n^m x_k|$.

II. THE RIESZ SEQUENCE SPACE $r^q(\mathcal{M}, \Delta_v^m, u, p, s)$ OF NON-ABSOLUTE TYPE.

Let $\mathcal{M} = (M_i)$ be Musielak-Orlicz function, $u = (u_i)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then we defined new difference sequence space $r^q(\mathcal{M}, \Delta_v^m, u, p, s)$ as follows:

$$r^q(\mathcal{M}, \Delta_v^m, u, p, s) = \left\{x = (x_k) \in \omega : \sum_k |\frac{1}{Q_k^{s+1}} \sum_{i=0}^{k} M_i\left(|u_i q_i \Delta_v^m x_i|\right)|^{p_k} < \infty\right\},$$

Where $0 < p_k \le H < \infty$.

Which is a generalization of space defined and studied by Raj and Anand [15].

With the definition of matrix domain (1.1), the sequence space $r^q(\mathcal{M}, \Delta_v^m, u, p, s)$ may be redefined as

$$r^q(\mathcal{M}, \Delta_v^m, u, p, s) = \{l(p)\}_{R^q(\mathcal{M}, \Delta_v^m, u, s)}$$

Where $r^q(\mathcal{M}, \Delta_v^m, u, s)$ denotes the matrix $R^q(\mathcal{M}, \Delta_v^m, u, p, s) = r_{nk}^q(\mathcal{M}, \Delta_v^m, u, s)$ defined by

$$r_{nk}^{q}(\mathcal{M},\Delta_v^m,u,s)=\begin{cases}\frac{1}{Q_n^{s+1}}\left(M_k\left(u_kq_k-M_{k+1}(u_{k+1}q_{k+1}\right)\right) & if\ 0\le k\le n-1\\ \frac{M_n(u_nq_n\ )}{Q_n^{s+1}} & if\ k=n\\ 0 & if\ k>n\end{cases}$$

Define the sequence $y=(y_k)$ which will be used by the $R^q(\mathcal{M},\Delta_v^m,u,s)$-transform of a sequence $x=(x_k)$, we have

$$y_k=\frac{1}{Q_k^{s+1}}\sum_{i=1}^{k}M_i\left(|u_iq_i\Delta_v^m x_i|\right) \qquad (2.1)$$

The main purpose of this paper is to study some generalized difference sequence space derived by Riesz mean and Musielak-Orlicz function. We shall show that these spaces are complete and paranormed spaces we also determined the $\alpha-,\beta-,and\ \gamma-duals$ of these spaces.

Theorem 2.1 Let $\mathcal{M}=(M_j)$ be Musielak-Orlicz function, $u=(u_j)$ a sequence of strictly positive real numbers and $p=(p_k)$ be bounded sequence of positive real numbers. Then $r^q(\mathcal{M},\Delta_v^m,u,p,s)$ is a complete linear metric space paranormed by

$$g(x)=\left[\sum_k\left|\frac{1}{Q_k^{s+1}}\sum_{j=0}^{k-1}\left(M_j\left(u_jq_j\right)-M_{j+1}\left(u_{j+1}q_{j+1}\right)\right)x_j+\frac{M_k(u_kq_k)x_k}{Q_k^{s+1}}\right|^{p_k}\right]^{1/H}$$

With $0<p_k\le D<\infty. and\ H=\max\{1,D\}$

Proof . the linearity of $r^q(\mathcal{M},\Delta_v^m,u,p,s)$ follows from the inequality. See[1]. For $x,y\in r^q(\mathcal{M},\Delta_v^m,u,p,s)$.

$$\left[\sum_k\left|\frac{1}{Q_k^{s+1}}\sum_{j=0}^{k-1}\left(M_j\left(u_jq_j\right)-M_{j+1}\left(u_{j+1}q_{j+1}\right)\right)(x_j+y_j)+\frac{M_k(u_kq_k)}{Q_k^{s+1}}(x_j+y_j)\right|^{p_k}\right]^{1/H} \qquad (2.2)$$

$$\le\left[\sum_k\left|\frac{1}{Q_k^{s+1}}\sum_{j=0}^{k-1}\left(M_j\left(u_jq_j\right)-M_{j+1}\left(u_{j+1}q_{j+1}\right)\right)x_j+\frac{M_k(u_kq_k)x_k}{Q_k^{s+1}}\right|^{p_k}\right]^{1/H}$$

$$+\left[\sum_k\left|\frac{1}{Q_k^{s+1}}\sum_{j=0}^{k-1}\left(M_j\left(u_jq_j\right)-M_{j+1}\left(u_{j+1}q_{j+1}\right)\right)y_j+\frac{M_k(u_kq_k)}{Q_k^{s+1}}y_k\right|^{p_k}\right]^{1/H}$$

For any $\alpha\in\mathbb{R}$ (see [14])

$$|\alpha|^{p_k}\le\max(1,|\alpha|^H). \qquad (2.3)$$

It is clear that $g(\theta)=0\ and\ g(-x)=g(x), for\ all\ x\in r^q(\mathcal{M},\Delta_v^m,u,p,s)$. Again the inequality (2.2) and (2.3) yield the sub additivity of g

$$g(\alpha x)\le\max(1,|\alpha|)\,g(x).$$

Let $\{x^n\}$ be any sequence of points of the space$r^q(\mathcal{M},\Delta_v^m,u,p,s)$ such that $g(x^n-x)\to 0$ and (α_n) is a sequence of scalars such that $\alpha_n\to\alpha$. Then since the inequality

$$g(x^n)\le g(x)+g(x^n-x)$$

Holds by subadditivity of $g,\{g(x^n)\}$ is bounded and we thus have

$$g(\alpha_nx^n-\alpha x)=\left[\sum_k\left|\frac{1}{Q_k^{s+1}}\sum_{j=0}^{k-1}\left(M_j\left(u_jq_j\right)-M_{j+1}\left(u_{j+1}q_{j+1}\right)\right)(\alpha_nx_j^n-\alpha x_j)+\frac{M_k(u_kq_k)x_k}{Q_k^{s+1}}\right|^{p_k}\right]^{1/H}$$

$$\le|\alpha_n-\alpha|^{1/H}g(x^n)+|\alpha|^{1/H}g(x^n-x).$$

Which tends to zero as $n\to\infty$. This proves that the scalar multiplication is continuous. Hence g is paranorm on the $r^q(\mathcal{M},\Delta_v^m,u,p,s)$.

Now we prove the completeness of $r^q(\mathcal{M},\Delta_v^m,u,p,s)$.

Let $\{x^j\}$ be any Cauchy sequence in the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$. Where $x^j=(x_0^j,x_1^j,\ldots\}$. Then for any $\epsilon>0$, there exists a positive integer $n_0(\epsilon)$ such that

$$g(x^i-x^j)<\epsilon, for\ all\ i,j\ge n_0(\epsilon) \qquad (2.4)$$

Using definition of g for each fixed $k\in\mathbb{N}$ that

$$|(R^q(\mathcal{M},\Delta_v^m,u,s)x^i)_k-(R^q(\mathcal{M},\Delta_v^m,u,s)x^j)_k|$$

$$\le[\sum_k|(R^q(\mathcal{M},\Delta_v^m,u,s)x^i)_k-(R^q(\mathcal{M},\Delta_v^m,u,s)x^j)_k|^{p_k}]^{1/H}<\epsilon$$

For $i,j\ge n_0(\epsilon)$

Which yields that $\{(R^q(\mathcal{M},\Delta_v^m,u,s)x^0)_k,(R^q(\mathcal{M},\Delta_v^m,u,s)x^1)_k,\ldots\}$ is Cauchy sequence of real numbers for every fixed $k\in\mathbb{N}$. Since R is complete, it converges say

$(R^q(\mathcal{M},\Delta_v^m,u,s)x^i)_k\to(R^q(\mathcal{M},\Delta_v^m,u,s)x)_k as\ j\to\infty,$

Using these infinitely many limits $(R^q(\mathcal{M},\Delta_v^m,u,s)x)_0,(R^q(\mathcal{M},\Delta_v^m,u,s)x)_1,\ldots$

We definite the sequence $\{ (R^q(\mathcal{M},\Delta_v^m,u,s)x)_0, (R^q(\mathcal{M},\Delta_v^m,u,s)x)_1, \ldots\}$
From (2.4) for each $t \in \mathbb{N}$ and $i,j \geq n_0(\epsilon)$

$$\sum_{k=0}^{t} |(R^q(\mathcal{M},\Delta_v^m,u,s)x^i)_k - (R^q(\mathcal{M},\Delta_v^m,u,s)x^j)_k|^{p_k} \leq g(x^i - x^j)^H < \epsilon^H \tag{2.5}$$

Take any $i,j \geq n_0(\epsilon)$. first , let $j \to \infty$ in (2.5) and then $t \to \infty$
We obtain, $g(x^i - x) \leq \epsilon$.
Finally , taking $\epsilon = 1$ in (2.5) and letting $j \geq n_0(1)$. We have by minkowski's inequality for each $t \in \mathbb{N}$ that

$$[\sum_{k=0}^{t} |(R^q(\mathcal{M},\Delta_v^m,u,s)x)_k|^{p_k}]^{1/H} \leq g(x^i - x) + g(x^i).$$
$$\leq 1 + (x^i)$$

Which implies that $x \in r^q(\mathcal{M},\Delta_v^m,u,p,s)$. Since $g(x - x^i) \leq \epsilon for\ all\ i \geq n_0(\epsilon)$ it follows that $x^i \to x\ as\ i \to \infty$. Hence the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$ is complete.

Theorem 2.2 Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function $u = (u_i)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of real numbers. Then the sequence space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$ of non absolute type is linearly isomorphic to the $\ell(p)$ where $0 < p\_k \leq D < \infty$.

Proof. To show that the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$ and $\ell(p)$ are isomorphic, we have to show that there exists a linear bijection between these spaces. Define a linear transformation
$T: r^q(\mathcal{M},\Delta_v^m,u,p,s) \to \ell(p)\ by\ x \to y = Tx$ by using equation (2.2).The linearity of T is trivial. Further it is obvious that $x = \theta$, whenever $T(x) = T(\theta)$ and hence T is injective. Let $y \in \ell(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1} \left(\frac{1}{M_n(u_n q_n)} - \frac{1}{M_{n+1}(u_{n+1}q_{n+1})} Q_k^{s+1} y_k + \frac{Q_k^{s+1}}{M_k(u_k q_k)} y_k\right) for\ all\ k \in \mathbb{N}.$$

Then

$$g(x) = \left[\sum_k |\frac{1}{Q_k^{s+1}} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1}q_{j+1})x_j + \frac{M_k(u_k q_k)x_k}{Q_k^{s+1}}|^{p_k}\right]^{1/H}$$
$$= \left[\sum_k \left|\sum_{j=0}^{k} \delta_{kj} y_j\right|^{p_k}\right]^{1/H}$$
$$= \left[\sum_k |y_k|^{p_k}\right]^{1/H}$$
$$= g_1(y) < \infty$$

Where $\delta_{kj} = \begin{cases} 1\ , & if\ k = j \\ 0\ , & if\ k \neq j \end{cases}$

Thus , we have $x \in r^q(\mathcal{M},\Delta_v^m,u,p,s)$. Consequently T is surjective and paranorm preserving. Hence , T is linear bijection and this shows the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$ and $\ell(p)$ are linearly isormorphic.

III. BASIS AND $\alpha-, \beta-, AND\ \gamma - DUALS\ OF\ THE\ SPACE\ r^q(\mathcal{M},\Delta_v^m,u,p,s)$.

In this section, we compute $\alpha-, \beta - and\ \gamma - duals$ of the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$ and finally we give the basis for the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$.
For the sequence space X and Y, define the set
$S(X,Y) = \{z = (z_k): xz = (x_k z_k) \in Y\}$
The $\alpha-, \beta - and\ \gamma - duals$ of the sequence space X, respectively denoted by $X^\alpha, X^\beta\ and\ X^\gamma$ which are defined by

$$X^\alpha = S(X,\ell_1),\ \ X^\beta = S(X,cs)\ \ and\ X^\gamma = S(X,bs)$$

Firstly, we state some lemmas which are required in proving our theorems.

Lemma 3.1([12]) (i) Let $1 < p_k \leq D < \infty$. Then $A \in (\ell(p),\ell_1)$ if and only if there exists an integer $B > 1$ such that

$$\sup_{k \in \mathcal{F}} \sum_k |\sum_{n \in \mathbb{N}} a_{nk}\ B^{-1}|^{p_k'} < \infty.$$

(ii) Let $0 < p_k \leq 1$. Then $A \in (\ell(p),\ell_1)$ if and only if

$$\sup_{k \in \mathcal{F}} \sup_k |\sum_{n \in \mathbb{N}} a_{nk} B^{-1}|^{p_k} < \infty.$$

Lemma 3.2 [13] (i) Let $1 < p_k \leq D < \infty$. Then $A \in (\ell(p), \ell_\infty)$ if and only if there exists an integer $B > 1$ such that

$\sup_n \sum_k |\alpha_{nk} B^{-1}|^{p_k'} < \infty.$ (3.1)

(ii) $1 < p_k \leq 1$, for every $k \in \mathbb{N}$. Then $A \in (\ell(p), \ell_\infty)$ if and only if

$\sup_{n,k} |\alpha_{nk}|^{p_k} < \infty$ (3.2)

Lemma 3.3 [10] Let $1 < p_k \leq D < \infty$. for every $k \in \mathbb{N}$. Then $A \in (\ell(p), c)$ if and only if (3.1), (3.2) hold along with

$\lim_n \alpha_{nk} = \beta_k \; for \; k \in \mathbb{N}$. also holds (3.3)

Theorem 3.1. Let $\mathcal{M} = (M_j)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Define the sets $D_1(\mathcal{M}, \Delta_v^m, u, p, s)$ and $D_2(\mathcal{M}, \Delta_v^m, u, p, s)$ as follows:

$$D_1(\mathcal{M}, \Delta_v^m, u, p, s) = \cup_{B>1}\{\alpha = (\alpha_k)$$

$$\in \omega\colon \sup_{k\in\mathcal{F}} \sum_k |\sum_{n\in k}[(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})})Q_k^{s+1}\alpha_n + \frac{Q_n^{s+1}}{M_n(u_n q_n)}\alpha_n]B^{-1}|^{p_k'} < \infty$$

And

$D_2(\mathcal{M}, \Delta_v^m, u, p, s) =$

$\cup_{B>1}\left\{\alpha = (\alpha_k) \in \omega\colon \sum_k |[(\frac{\alpha_k}{M_k(u_k q_k)} + \left(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^n \alpha_i)Q_k^{s+1}]B^{-1}|^{p_k'} < \infty\right\}.$

Then

$[r^q(\mathcal{M}, \Delta_v^m, u, p, s)]^\alpha = D_1(\mathcal{M}, \Delta_v^m, u, p, s)$ and

$[r^q(\mathcal{M}, \Delta_v^m, u, p, s)]^\beta = D_2(\mathcal{M}, \Delta_v^m, u, p, s) \cap cs.$

Proof. Let us take any $\alpha = (\alpha_k) \in \omega$ we can easily derive with (2.1) that

$$\alpha_n x_n = \sum_{k=0}^{n-1}\left(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\alpha_n Q_k^{s+1} y_k + \frac{\alpha_n}{M_n(u_n q_n)} Q_n^{s+1} y_n$$

$= (cy)_n.$ (3.4)

Where $c = (c_{nk})$ is defined as

$$c_{nk} = \begin{cases} \left(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\alpha_n Q_k^{s+1}, & if \; 0 \leq k \leq n-1 \\ \frac{\alpha_n}{M_n(u_n q_n)} Q_n^{s+1} & if \; k = n \\ 0 & if \; k > n \end{cases}$$

For all $n, k \in \mathbb{N}$. Thus , we observe by combining (3.4) with (i) of lemma (3.1) that $\alpha x = (\alpha_n x_n) \in \ell_1 \; whenever \; x = (x_n) \in r^q(\mathcal{M}, \Delta_v^m, u, p, s)$ if and only if $cy \in \ell_1 \; whenever \; y \in \ell_p$. This gives the result that $[r^q(\mathcal{M}, \Delta_v^m, u, p, s)]^\alpha = D_1(\mathcal{M}, \Delta_v^m, u, p, s)$

Further, consider the equation

$$\sum_{k=0}^n \alpha_{nk} x_k = \sum_{k=0}^n \left[\left(\frac{\alpha_k}{M_k(u_k q_k)} + \left(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^n \alpha_i\right) Q_k^{s+1}\right] y_k \quad (3.5)$$

$$= (Dy)_n$$

Where $D = (d_{nk})$ is define as

$$d_{nk} = \begin{cases} \left(\frac{\alpha_k}{M_k(u_k q_k)} + \left(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^n \alpha_i\right) Q_k^{s+1} & if \; 0 \leq k \leq n \\ 0 & if \; k > n \end{cases}$$

Thus , we deduce from Lemma (3.3) with (3.5) that $\alpha x = (\alpha_n x_n) \in cs$ whenever $x = (x_n) \in r^q(\mathcal{M}, \Delta_v^m, u, p, s)$ if and only if $Dy \in c \in whenever \; y \in \ell(p)$. Therefore , we derive from (3.1) that

$$\sum_k |\left[\left(\frac{\alpha_k}{M_k(u_k q_k)} + \left(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^n \alpha_i\right) Q_k^{s+1}\right] B^{-1}|^{p_k'} < \infty \quad (3.6)$$

And the $\lim_n d_{nk}$ exists and hence shows that $[r^q(\mathcal{M}, \Delta_v^m, u, p, s).]^\beta = D_2(\mathcal{M}, \Delta_v^m, u, p, s) \cap cs$.

From Lemma (3.2) together with (3.5) that $\alpha x = (\alpha_k x_k) \in bs$ whenever $x \in r^q(\mathcal{M}, \Delta_v^m, u, p, s)$ if and only if $Dy \in \ell_\infty$ whenever $y = (y_k) \in \ell(p)$. Therefore , we again obtain the condition (3.6) which means that $[r^q(\mathcal{M}, \Delta_v^m, u, p, s).]^\gamma = D_2(\mathcal{M}, \Delta_v^m, u, p, s) \cap cs$. And the proof of the theorem is complete.

Theorem 3.2 Let $1 < p_k \leq 1$ for any $k \in \mathbb{N}$ $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Define the sets $D_3(\mathcal{M}, \Delta_v^m, u, p, s)$ and $D_4(\mathcal{M}, \Delta_v^m, u, p, s)$ as follows:

$$D_3(\mathcal{M},\Delta_v^m,u,p,s) = \Big\{\alpha=(\alpha_k) \in \omega:\ \sup_{k\in\mathcal{F}}\sup_k |\sum_{n\in k}\Big[\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\alpha_n Q_k^{s+1}+\frac{\alpha_n}{M_n(u_nq_n)}Q_n^{s+1}\Big]|^{p_k}<\infty\Big\}$$

And

$$D_4(\mathcal{M},\Delta_v^m,u,p,s) = \Big\{\alpha=(\alpha_k) \in \omega:\ \sup_k \Big|\Big[\Big(\frac{\alpha_k}{M_k(u_kq_k)}+\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\sum_{i=k+1}^{n}\alpha_i\Big)Q_k^{s+1}\Big]\Big|^{p_k}<\infty\Big\}$$

$$[r^q(\mathcal{M},\Delta_v^m,u,p,s).]^{\alpha} = D_3(\mathcal{M},\Delta_v^m,u,p,s)$$
$$[r^q(\mathcal{M},\Delta_v^m,u,p,s).]^{\beta} = D_4(\mathcal{M},\Delta_v^m,u,p,s)$$

Proof . This is obtained by proceeding in the proof of theorem (3.1) by using second parts of Lemma (3.1),(3.2) and (3.3) instead of the first parts, so we omit the details.

Schauder basis for the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$will be given

Theorem 3.3 Let $\mathcal{M}=(M_j)$ be Musielak-Orlicz function, $u=(u_k)$ be a sequence of strictly positive real numbers and $p=(p_k)$ be a bounded sequence of real numbers. Define the sequence $b^{(k)}(q)=\{b_n^{(k)}(q)\}$ of the element of the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$, for every fixed $k\in\mathbb{N}$

$$b_n^{(k)}(q)=\begin{cases}\Big(\frac{1}{M_n(u_nq_n)}-\frac{1}{M_{n+1}(u_{n+1}q_{n+1})}\Big)Q_n^{s+1}+u_n^{-1}\frac{Q_k^{s+1}}{M_k(u_kq_k)}, & if\ 0\le n\le k-1\\ 0 & if\ n>k-1\end{cases}$$

Then the sequence $\{b_n^{(k)}(q)\}$ is the basis for the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$ any element in the space has a unique representation of the form

$$x=\sum_k\lambda_k(q)b^{(k)}(q) \tag{3.7}$$

Where $\lambda_k(q)=(R^q(\mathcal{M},\Delta_v^m,u,s)x)_k\ for\ all\ k\in\mathbb{N}1<p_k\le D<\infty$.

Proof . it is clear that $\{b^{(k)}(q)\}\subset r^q(\mathcal{M},\Delta_v^m,u,p,s)$, since

$$R^q(\mathcal{M},\Delta_v^m,u,s)b^{(k)}(q)=e^{(k)}\in\ell(p)for\ \ k\in\mathbb{N} \tag{3.8}$$

and $1<p_k\le D<\infty\ where\ \ e^{(k)}$ is a sequence whose only non zero term is 1 in k^{th} place for each $k\in\mathbb{N}$.

Let $x\in r^q(\mathcal{M},\Delta_v^m,u,p,s)$ be given. For every non-negative integer t, we put

$$x^{(t)}=\sum_{k=0}^t\lambda_k(q)b^{(k)} \tag{3.9}$$

Then, we obtain by applying $R^q(\mathcal{M},\Delta_v^m,u,s)$ to (3.9) with (3.8) that

$$R^q(\mathcal{M},\Delta_v^m,u,s)x^{[t]}=\sum_{k=0}^t\lambda_k(q)R^q(\mathcal{M},\Delta_v^m,u,s)b^{(k)}(q)=\sum_{k=0}^tR^q(\mathcal{M},\Delta_v^m,u,s)x)_ke^{(k)}.$$

And

$$(R^q(\mathcal{M},\Delta_v^m,u,s)(x-x^{[t]}))=\begin{cases}0 & if\ 0\le i\le t\\ R^q(\mathcal{M},\Delta_v^m,u,s)x)_i & if\ i>t.\end{cases}$$

Hence

$$g(x-x^{[t]})=\Big(\sum_{i=t}^{\infty}|R^q(\mathcal{M},\Delta_v^m,u,s)x)_i|^{p_k}\Big)^{1/H} \le\Big(\sum_{i=t_0}^{\infty}|R^q(\mathcal{M},\Delta_v^m,u,s)x)_i|^{p_k}\Big)^{1/H} <\frac{\epsilon}{2} <\epsilon$$

For all $t>t_0$ which proves that $x\in r^q(\mathcal{M},\Delta_v^m,u,p,s)$ is represented as equation (3.7).

Let us show that the uniqueness of the representation for $x\in r^q(\mathcal{M},\Delta_v^m,u,p,s)$ given by (3.7)

Suppose on contrary that there exists a representation $x=\sum_k\mu_k(q)b^{(k)}(q)$. since the linear transformation T from $x\in r^q(\mathcal{M},\Delta_v^m,u,p,s)$ to $\ell(p)$ used in Theorem 2.2 is continuous, we have

$$(R^q(\mathcal{M},\Delta_v^m,u,s)x)_n = \sum_k \mu_n(q) R^q(\mathcal{M},\Delta_v^m,u,s)b^{(k)}(q)_n = \sum_k \mu_k(q)e^{(k)} = \mu_n(q)$$

For all $n \in \mathbb{N}$ which contradicts the fact that $R^q(\mathcal{M},\Delta_v^m,u,s)x)_n = \lambda_n(q), \forall n \in \mathbb{N}$
Hence the representation (3.7) is unique.

IV. CONCLUSION

We observe that the space $r^q(\mathcal{M},\Delta_v^m,u,p,s)$ is not only linear but also a complete linear metric space paranormed by

$$g(x) = \left[\sum_k |\frac{1}{Q_k^{s+1}}\sum_{j=0}^{k-1}\left(M_j\left(u_j q_j\right) - M_{j+1}\left(u_{j+1}q_{j+1}\right)\right)x_j + \frac{M_k(u_k q_k)x_k}{Q_k^{s+1}}|^{p_k}\right]^{1/H}$$

With, $0 < p_k \leq D < \infty, and\ H = \max\{1, D\}$. The space is also isomorphic to the space of p- summable sequences $\ell(p)$.

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