International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 5 Issue 2 || February. 2017 || PP-01-03

The Total Strong Split Domination Number of Graphs

T.Nicholas, T.Sheeba Helen
Department of Mathematics, St.Judes College, Thoothoor, TamilNadu, India.
Department of Mathematics, Holy Cross College (Autonomous), Nagercoil, TamilNadu, India.

ABSTRACT: *A total dominating set D of graph G = (V, E) is a total strong split dominating set if the induced subgraph < V-D > is totally disconnected with atleast two vertices. The total strong split domination number $\gamma_{tss}(G)$ is the minimum cardinality of a total strong split dominating set. In this paper, we characterize total strong split dominating sets and obtain the exact values of $\gamma_{tss}(G)$ for some graphs. Also some inequalities of $\gamma_{tss}(G)$ are established.*
KEYWORDS: *Domination number, split domination number, total domination number, strong split domination number, total strong split domination number.*
Subject Classification: *05C*

I. INTRODUCTION

The graphs considered here are finite, undirected, without loops, multiple edges and have atmost one component which is not complete. For all graph theoretic terminology not defined here, the reader is referred to [2]. A set of vertices D in a graph G is a dominating set, if every vertex in V-D is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. This is well studied parameter as we can see from [3]. For a comprehensive introduction to theoretical and applied facts of domination in graphs the reader is directed to the book [3]. A dominating set D of a graph G is a total dominating set if the induced sub graph < D > has no isolated vertices. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G.This concept was introduced by Cockayne, Dawes and Hedetniemi in [1].

V. R. Kulli and B. Janakiram introduced the concept of split domination in [6]. A dominating set D of a graph G = (V, E) is a split dominating set if the induced subgraph < V-D > is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split dominating set.

Strong split domination was introduced by V. R. Kulli and B. Janakiram in [7]. A dominating set D of a graph G = (V, E) is a strong split dominating set if the induced subgraph < V-D > is totally disconnected with atleast two vertices. The strong split domination number $\gamma_{ss}(G)$ is the minimum cardinality of a strong split dominating set.

A total dominating set D of a connected graph G is a total split dominating set if the induced subgraph < V-D > is disconnected. The total split domination number $\gamma_{ts}(G)$ is the minimum cardinality of a strong split dominating set. This concept was introduced by B. Janakiram, Soner and Chaluvaraju in [5].

We introduce a new concept namely total strong split domination number. A total dominating set D of a connected graph G is a total strong split dominating set if the induced subgraph < V-D > is totally disconnected with atleast two vertices. The total strong split domination number $\gamma_{tss}(G)$ is the minimum cardinality of a total strong split dominating set.

II. RESULTS

Theorem 1. For any graph G , $\gamma_s(G) \leq \gamma_{ss}(G) \leq \gamma_{tss}(G)$

Proof: This follows from the fact that every total strong split dominating set of G is a strong split dominating set of G and every strong split dominating set of G is a split dominating set of G.

The following two characterizations are easy to see, hence we omit their proofs.

Theorem 2. A total dominating set D of a graph G is a total strong split dominating set if and only if the following conditions are satisfied.

(i) V-D has atleast two vertices

(ii) For any vertices u, v $\in$ V-D, every u, v path contains atleast one vertex of D

Theorem 3. A total strong split dominating set of G is minimal if for each vertex $v \in D$,

there exists a vertex u $\in$ V-D such that u is adjacent to v.

We now consider a lower bound on $\gamma_{tss}(G)$ in terms of the minimum degree, the order and the size of G.

Theorem 4. If G has no isolated vertices and p $\geq$ 3, then $p - \lceil q/\delta(G) \rceil \leq \gamma_{tss}(G)$,

where $\delta(G)$ is the minimum degree of G.

Proof:
Let D be a γ_{tss}-set of G.Then every vertex in V-D is adjacent with atleast $\delta(G)$ vertices in D. This implies that q $\geq |V-D|\ \delta(G)$.
Thus the theorem follows from the fact that $\lceil q/\delta(G)\rceil \geq q/\delta(G)$.
Theorem 5. For any connected graph G, $\gamma_{tss}(G) = \alpha_0(G)$, where $\alpha_0(G)$ is the vertex covering number.
Proof: Let S be the minimum independent set of vertices in G. Then V-S is the total strong split dominating set of G. v $\in$ V-S, then there exists a vertex u $\in$ S such that v is adjacent to u.Then by Theorem 3, V-S is minimal and is the minimal covering for G. $|V-S| = \alpha_0(G)$. This proves the theorem.
Corollary 5.1. For any graph G, $\gamma_{tss}(G) \leq q$
Proof: By (3) $\gamma_{tss}(G) = \alpha_0(G)$
$= p - \beta_0(G)$
Where $\beta_0(G)$ is the independence number of G.
Thus the result follows from the fact that $\beta_0(G) \geq \gamma(G) \geq p - q$.
Now we list the exact values of $\gamma_{tss}(G)$ for some standard graphs.
Proposition 6.1 For any cycle C_n with n $\geq$ 6 vertices

$$\gamma_{tss}(C_n) = \begin{cases} \frac{2n}{3} & if\ n \equiv 0\ (mod\ 3) \\ \frac{2n+1}{3} & if\ n \equiv 1\ (mod\ 3) \\ \frac{2}{3}(n+1) & if\ n \equiv 2\ (mod\ 3) \end{cases}$$

Proof: Let $V(C_n) = \{v_0, v_1, v_2, v_3, \ldots, v_{n-1}\}$ be the vertex set of the cycle C_n. Let D be the total strong split dominating set of C_n. Consider the sets,
$D_1 = \{ v_{3i}, v_{3i+1} / i = 0, 1, 2, \ldots, \frac{n-3}{3}\}$ when $n \equiv 0(mod\ 3)$.
$D_2 = \{ v_{3i}, v_{3i+1} / i = 0, 1, 2, \ldots, \frac{n-1}{3}\} \cup \{ v_{n-2} \}$ when $n \equiv 1(mod\ 3)$.
$D_3 = \{ v_{3i}, v_{3i+1} / i = 0, 1, 2, \ldots, \frac{n-2}{3}\} \cup \{ v_{n-3}, v_{n-2} \}$ when $n \equiv 2(mod\ 3)$.
The above three sets achieve the total strong split property of C_n in the respective parity conditions.
Corollary 6.2 Let G be a Hamiltonian graph on n vertices. Then $\gamma_{tss}(G) \geq \gamma_{tss}(C_n)$

Proposition 6.3 For any path P_n with n $\geq$ 4 vertices

$$\gamma_{tss}(P_n) = \begin{cases} \frac{2n}{3} & if\ n \equiv 0\ (mod\ 3) \\ \frac{2}{3}(n-1) & if\ n \equiv 1\ (mod\ 3) \\ \frac{2n-1}{3} & if\ n \equiv 2\ (mod\ 3) \end{cases}$$

Proof: Let P_n be the path of order n. $V(P_n) = \{ v_0, v_1, v_2, v_3, \ldots, v_{n-1}\}$. Let D be the total strong split dominating set of P_n. When n = 4, $V(P_4) = \{ v_0, v_1, v_2, v_3 \}$.Then $D = \{ v_1, v_2 \}$.
When n = 5, $V(P_5) = \{ v_0, v_1, v_2, v_3, v_4 \}$.Then $D = \{ v_1, v_2, v_3 \}$.
If n $\equiv$ 0 (mod 3) then D contains v_{3i+1}, v_{3i+2} where i = 0, 1, . . . , $\frac{n-3}{3}$.
$D = \{ v_1, v_2, v_4, v_5, \ldots, v_{n-2}, v_{n-1} \}$. Hence $|D| = 2\left(\frac{n-3}{3} + 1\right) = \frac{2n}{3}$
If n $\equiv$ 1 (mod 3) then D contains v_{3i+1}, v_{3i+2} where i = 0, 1, . . . , $\frac{n-4}{3}$.
$D = \{ v_1, v_2, v_4, v_5, \ldots, v_{n-3}, v_{n-2} \}$. Thus $|D| = 2\left(\frac{n-4}{3} + 1\right) = \frac{2}{3}(n-1)$
If n $\equiv$ 2 (mod 3) then D has v_{3i+1}, v_{3i+2} where i = 0, 1, . . . , $\frac{n-5}{3}$ and also v_{n-2}.
$D = \{ v_1, v_2, v_4, v_5, \ldots, v_{n-4}, v_{n-3}, v_{n-2} \}$. In this case $|D| = 2\left(\frac{n-5}{3} + 1\right) + 1 = \frac{2n-1}{3}$
Hence the result follows. ∎

Proposition 6.4 For any wheel W_n with n $\geq$ 6 vertices, $\gamma_{tss}(W_n) = \gamma_{tss}(C_n) + 1$
Proof: Let $V(W_n) = \{v_0, v_1, v_2, v_3, \ldots, v_{n-1}\}$ be the vertex set of the wheel W_n. Any minimal total strong split dominating set must contain the apex vertex v_0 since otherwise the induced graph
$< V- D >$ will have an edge v_0v_i for some $v_i \notin D$ which is a contradiction to the strong split condition. Hence $D = D_1 \cup \{v_0\}$ where D_1 is any minimal total strong split dominating set.
Proposition 6.5 For any complete bipartite graph $K_{m,n}$, where $2 \leq m < n$, $\gamma_{tss}(K_{m,n}) = m+1$

Proof: Let V_1 and V_2 be the partite sets of the complete bipartite graph $K_{m,n}$, where $2 \leq m < n$ and $|V_1|$=m , $|V_2|$=n. Let D be the minimal total strong split set of $K_{m,n}$. Since < V- D > is totally disconnected D must contain all the vertices of one partite set. As D is minimal D contains every vertex of V_1. Since < V_1 > contains isolated vertices, D must have any one vertex from the partition V_2 so that < $V_1 \cup \{v\}$ > does not contain any isolated vertex for any $v \in V_2$. Hence D = < $V_1 \cup \{v\}$ > where $v \in V_2$ is the minimal total strong split dominating set of $K_{m,n}$.

Corollary 6.6 For any Star $K_{1,n}$ with $n \geq 4$ vertices, $\gamma_{tss}(K_{1,n}) = 2$.

The double star $S_{m,n}$ is the graph obtained from the joining centers of $K_{1,m}$ and $K_{1,n}$ with an edge. The centers of $K_{1,m}$ and $K_{1,n}$ are called central vertices of $S_{m,n}$

Proposition 6.7 If $S_{m,n}$, $2 \leq m \leq n$ is double star, then $\gamma_{tss}(S_{m,n}) = 2$

Proof: Let $S_{m,n}$ be a double star where $1 \leq m \leq n$. Then $S_{m,n}$ has a total strong split dominating set D containing the central vertices of $S_{m,n}$. Thus the induced subgraph < V- D > contains m+n isolated vertices. Thus < V- D > is totally disconnected and < D > has no isolated vertices. Hence the theorem holds.

The crown graph $C_p \odot K_1$ is the graph obtained from cycle C_p by attaching a pendant edge to each vertex of the cycle.

Proposition 6.8 $\gamma_{tss}(C_p \odot K_1) = p$, where p is the length of the cycle C_p.

Proof: Let G = $C_p \odot K_1$. Let D be the minimal total strong split dominating set.

$V(G) = \{ v_0, v_1, v_2, \ldots, v_{n-1}\} \cup \{ u_0, u_1, u_2, \ldots, u_{n-1} \}$.

$E(G) = \{ v_i v_{i+1} / i = 0, 1, 2, \ldots, n-1$, subscript modulo n$\} \cup \{ u_i v_i / i = 0, 1, 2, \ldots, n-1\}$. D must have all the cycle vertices, for otherwise < V-D > does not become totally disconnected. Hence D serves as a total dominating set, which preserves the strong split prpperty. Infact, D is the unique γ_{tss} – set of G.

Theorem 7. For a tree $T \neq K_{1,n}$, $\gamma_{tss}(T) \leq n - |L|$,

where $|L|$ denotes the pendant vertices of T.

Proof: Let $T \neq K_{1,n}$ be a tree with n vertices. Let L be the collection of all pendent vertices.

D be the minimal total strong split dominating set of T. D must contain all the support vertices of T. Let S be the set of support vertices. If $V(T) = S \cup L$, then D = S and $\gamma_{tss}(T) = n - |L|$.

If $V(T) \supsetneqq S \cup L$ then there exists a vertex $x \in V(T)$ such that $x \notin S \cup L$. In this case D must contain at least one such vertex x .Hence $|D| \geq |S|$. In other words, $\gamma_{tss}(T) \leq n - |L|$

Observation:

(i) For a tree $T \neq K_{1,n}$ vertices having degree equal to one does not belong to γ_{tss} –set.

(ii) In a tree vertices adjacent to pendant vertices belong to γ_{tss} –set.

(iii) Let $T \neq K_1$ contain only leaf and support vertices, then $\gamma_{tss}(T) = |S|$, where $|S|$ denotes the number of support vertices of T.

Theorem 8. If H is a connected spanning subgraph of G, then $\gamma_{tss}(G) \geq \gamma_{tss}(H)$

Proof: H be a connected spanning subgraph of a connected graph G. Let D' be the dominating set of H. W. l. g let D' be a total strong split dominating set of G also.

If every edge in G is incident with any vertex of D' in G then D' becomes the total strong split dominating set for G. If there exists an edge in G which is not incident with an vertex of D' , then there exists $D \supset D'$ in G such that D is a total strong split dominating set for G.

Theorem 9. If G is a connected graph with $n \geq 3$, then $\gamma_{tss}(G) \geq \gamma_c(G)$

Proof: Let D be a γ_{tss} –set of G. Let D' be the γ_c –set of G. By definition < D' > is connected.

The induced subgraph < V- D' > contains isolated vertices and connected components. Let

$u_i \in D'$. $N(u_i)$ either belongs to D or V-D. Hence $|D| \geq |D'|$. Therefore $\gamma_{tss}(G) \geq \gamma_c(G)$.

Corollary 10: If G is a connected graph with $n \geq 3$ vertices then $\gamma_{tss}(G) \geq n - \varepsilon_T(G)$ where $\varepsilon_T(G)$ is the maximum number of pendant vertices in any spanning tree.

Proof: Let G be a connected graph .From [4] we know that $\gamma_c(G) = n - \varepsilon_T(G)$ for $n \geq 3$. By the above theorem we know that $\gamma_{tss}(G) \geq \gamma_c(G)$.Therefore $\gamma_{tss}(G) \geq n - \varepsilon_T(G)$.

Theorem 11. For any connected graph G with n > 6 vertices, $\gamma_{tss}(G) \leq n - 2$

Proof: Let G be a connected graph with n > 6 vertices. Let D be a γ_{tss} –set of G.

Choose a spanning tree T of G such that T contains minimum number of end vertices. Since every tree contains at least two pendant vertices u and v. The vertices u,v $\in$ D.
Hence $\gamma_{tss}(G) \leq n - 2$.

REFERENCES

[1]. C.J Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total Domination in Graphs*, Networks, 10 (1980) 211-219

[2]. Harary.F, Graph Theory, Addison-Wesley, Reading, MA, 1972.

[3]. T.W.Haynes, S.T.Hedetneimi, P.J.Slater, *Fundamentals of Domination in Graphs*, MarcelDekker, New York, 1988.

[4]. S.T. Hedetniemi and R.C. Laskar, Connected domination in graphs, In B. Bollobas,editor, Graph Theory and Combinatorics, Academic Press, London (1984) 209-218.

[5]. B.Janakiraman,N.D.Soner and B.Chaluvaraju, Total Split Domination in Graphs, Far East J.Appl. Math. 6(2002) 89-95

[6]. V. R. Kulli and B. Janakiram, The split domination number of a graph, *Graph Theory Notes of New York*. New York Academy of Sciences, XXXII, 16-19 (1997)

[7]. V. R. Kulli and B. Janakiram, *The strong split domination number of a graph*, Acta Ciencia Indica, Vol.XXXII M, No. 2 (2006) 715-720

[8]. E.Sampathkumar and H.B.Walikar, *The connected domination number of a graph*, J.Maths. Phys. Sci., 13 (1979) 607-613.