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Reflexive and Dihedral (Co) Homologies of $\square / 2$ -Graded Algebras

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ABSTRACT: *We are concerned with the dihedral (co)homology of a unital* $\square / 2$ *-graded algebra* A *over a field* K *with a graded involution and recall definitions and properties of Hochschild and cyclic (co) homology groups for a* $\square / 2$ *-graded algebra from [7]. A cyclic cohomology of algebras over the complex numbers* $\square$ *has given by Kastler. We introduce the reflexive and dihedral (co)homology groups for a* $\square / 2$ *-graded algebra* A *by defining the reflexive operator* r *and the dihedral operator* h *in the* $\square / 2$ *-graded case.*
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I. INTRODUCTION

Cyclic homology was discovered by Alain Connes, Joachim Cuntz, Daniel Quillen and Boris Tsygan in the early 1980's. Connes was looking for a target for the Chern character in the non-commutative setting, while Tsygan was motivated by the wish to have an additive version of algebraic K-theory. The (co)homology theories for algebras that we will need are known as cyclic type homologies. The theory for algebras is obtained by assigning to a given algebra a cyclic module. The group dihedral homology of an algebra over a field K with characteristic zero, i.e. $ch(K) = 0$ was introduced by Tsygan (1983). The dihedral homology is an important type of homology theory for discrete and operator algebra, we mean by discrete algebra an algebra over ring (algebra without topology) . In 1987 Loday [9], and Krassawkas, Lapin and Solovev [8] introduced and studied the dihedral (co)homology of involutive unital algebras. The authhors studied the dihedral (co)homology of certain classes of operator algebra [4],and [5]. The dihedral homology of algebras and its relation with quaternion homology has been studied by Loday [9]. The homology theory of some classes of C*-algebras has been studied in [1]. In this article we are concerned with the dihedral (co)homology of a unital $\square / 2$ -graded algebra A over a fixed field K of characteristic different from 2 , i.e. $ch(K) \neq 2$ with a graded involution. by defining the reflexive operator r and the dihedral operator h in the $\square / 2$ -graded case from [7].

Definition (1.1) :[6]
Let A be an associative unital algebra over a commutative ring K $(K = \square)$ with an involution $* : A \to A ; a \to a^*$ for all $a \in A$. Define the complex $C(A) = (C_n(A), b_n)$, where $C_n(A) = A^{\otimes(n+1)}, n \geq 0$ and $b_n : C_n(A) \to C_{n-1}(A)$ is the boundary operator

$$b_n(a_0 \otimes \ldots\ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} .$$

It is well known that $b_n b_{n+1} = 0$,and hence $Ker(b_n) \supset \operatorname{Im}(b_{n+1})$. The group

$$H_n(A) = H_n(C_*(A)) = \frac{Ker(b_n)}{\operatorname{Im}(b_{n+1})},$$

is called the Hochschild homology of algebra A , and denote by $HH_n(A)$. We act on the complex $C(A)$ by the cyclic group $\square_{n+1}$ of order $n+1$ by means of the cyclic operator $t_n : C_n(A) \to C_n(A)$ such that :

$$t_n(a_0 \otimes \ldots \otimes a_{n-1} \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1} \qquad (1.1) .$$

The quotient complex $CC_n(A) = \dfrac{C_n(A)}{\operatorname{Im}(1 - t_n)}$ is a subcomplex of the complex $C_n(A)$. Following [10] the cyclic homology of algebra A is the homology of the complex $CC_*(A)$, and denote by

$$HC_n(A) = H_n(CC_*(A), b_*) = H_n\left(\frac{C_*(A)}{\operatorname{Im}(1-t_n)}, b_*\right).$$

We act on the complex $C(A)$ by the reflexive group $\mathbb{Z}/2 = \{-1,+1\}$ of order 2 by means of the reflexive operator $r_n : C_n(A) \to C_n(A)$ such that:

$$r_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \alpha(-1)^{n(n+1)/2} a_0^* \otimes a_n^* \otimes \ldots \otimes a_1^* \qquad (1.2),$$

where $\alpha = \pm 1, \alpha^2 = 1, (r_n)^2 = 1$ and $a_i^* = \operatorname{Im}(a_i)$ under the involution $*$.

The quotient complex ${}^{\alpha}CR_n(A) = \dfrac{C_n(A)}{\operatorname{Im}(1-r_n)}$ is a subcomplex of the complex $C_n(A)$. Following [9] the reflexive homology of algebra A is the homology of the complex ${}^{\alpha}CR_*(A)$, and denote by

$${}^{\alpha}HR_n(A) = H_n(CR_*(A), b_*) = H_n\left(\frac{C_*(A)}{\operatorname{Im}(1-r_n)}, b_*\right).$$

If we act on the complex $C(A)$ by (1.1) and (1.2) together, we have the quotient complex ${}^{\alpha}CD_n(A) = \left(\dfrac{C_n(A)}{\operatorname{Im}(1-t_n) + \operatorname{Im}(1-r_n)}\right)$, which is a subcomplex of the complex $C_n(A)$. Following [5] the dihedral homology of algebra A is the homology of the complex ${}^{\alpha}CD_*(A)$, and denote by ;

$${}^{\alpha}HD_n(A) = H_n(CD_*(A), b_*) = H_n\left(\frac{C_n(A)}{\operatorname{Im}(1-t_n) + \operatorname{Im}(1-r_n)}, b_*\right).$$

Remarks: [6]

1- If we dropped the operators $t_n\,(r_n)$, we get the reflexive (cyclic) homology of algebra A, respectively.

2- There exists a long exact sequence :

$$\cdots \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S}$$
$$\ldots HC_{n-2}(A) \xrightarrow{B} HH_{q-1}(A) \xrightarrow{I} HC_{n-1}(A) \xrightarrow{S} \cdots$$

3- The long exact sequence :

$$\cdots \longrightarrow {}^{-\alpha}HD_n(A) \xrightarrow{\Delta} HC_n(A) \xrightarrow{\nabla} {}^{+\alpha}HD_n(A) \longrightarrow {}^{\alpha}HD_{n-1}(A) \longrightarrow \cdots$$

gives the short exact sequence :

$$0 \longrightarrow {}^{-\alpha}HD_n(A) \xrightarrow{\Delta} HC_n(A) \xrightarrow{\nabla} {}^{+\alpha}HD_n(A) \longrightarrow 0,$$

if Δ is connecting homeomorphism.

4- we have natural isomorphisms $(1/2 \in K)$

$$HC_n(A) \cong {}^{-\alpha}HD_n(A) \oplus {}^{+\alpha}HD_n(A).$$

The cohomology groups are defined analogously, see [8].

II. $\mathbb{Z}/2$ -GRADED ALGEBRAS WITH GRADED INVOLUTIONS.

In this section we introduce some basic concepts and facts concerning $\mathbb{Z}/2$ -graded algebras, see [7].

We set up the theory of $\mathbb{Z}/2$ -graded vector spaces, complexes and algebras. Let K be a field and $\alpha \in \{+,-\}$ which we identify with $\{+1,-1\}$.

Definition (2.1) :[7]
A $\mathbb{Z}/2$ -graded vector space is a K -vector space V equipped with an involution $\theta : V \to V$,defined by $x \to \alpha x$, $(x \in V$, $\alpha = \pm)$, such that, $\theta^2 = id_V$,that is, $\theta(x) = \alpha x$, $\alpha = \pm 1$, and $\theta^2(x) = x$. It is also, a $K[\mathbb{Z}/2]$ -module V .

Definition (2.2) :[7]
A positively graded complex of vector spaces

$$V_* = \left\{ \cdots \longrightarrow V_2 \xrightarrow{d} V_1 \xrightarrow{d} V_0 \longrightarrow 0 \right\}$$

is a $\mathbb{Z}/2$ -graded complex if all vector spaces V_i $(i \geq 0)$ are $\mathbb{Z}/2$ -graded and all differentials $d : V_i \to V_{i-1}$ are maps of $\mathbb{Z}/2$ -graded spaces.

Definition (2.3):[7]
A $\mathbb{Z}/2$ -graded algebra is an associative unital K -algebra A such that the multiplication is a map $\pi : A \otimes A \to A$ of $\mathbb{Z}/2$ -graded vector spaces.
That is :
(a) the involution $\theta : A \to A$ is a homomorphism of algebras : $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in A$, or
(b) $A^{\alpha} A^{\beta} \subset A^{\alpha\beta}$, $(\alpha, \beta \in \{\pm 1\})$.

We recall the definition and properties of Hochschild, and cyclic homology groups for $\mathbb{Z}/2$ -graded algebras from [7].
Let A be a $\mathbb{Z}/2$ -graded algebra .To A associate a cyclic $K[\mathbb{Z}/2]$ -module $C(A)$ defined by $C_n(A) = A^{\otimes(n+1)}$ .Face and degeneracy maps are given by:

$$d_i(a_0 \otimes \ldots\ldots \otimes a_n) = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & if \quad 0 \leq i < n \\ (-1)^{|a_n|(|a_0| + \ldots\ldots\ldots |a_{n-1}|)} a_n a_0 \otimes a_1 \otimes \ldots. \otimes a_{n-1} & if \quad i = n \end{cases} ,$$

$s_i(a_0 \otimes \ldots\ldots \otimes a_n) = a_0 \otimes \ldots. \otimes a_i \otimes 1 \otimes \ldots \otimes a_n$, for $0 \leq i \leq n$.
With the extra map

$$t(a_0 \otimes \ldots \otimes a_{n-1} \otimes a_n) = (-1)^{|a_n|(|a_0| + \ldots\ldots\ldots |a_{n-1}|)} a_n \otimes a_0 \otimes \ldots \otimes a_{n-1} .$$

The Hochschild homology groups $H_*(A, A)$ are by definition $H_*(C(A), b)$,
where $b = b_n : C_n(A) \to C_{n-1}(A)$ is the differential given by

$$b = b_n = \sum_{i=0}^{n} (-1)^i d_i = d_0 - d_1 + \ldots.. + (-1)^n d_n .$$

Note that, if A is trivially graded, then $H_*(A, A)$ coincide with the ones defined in the ungraded case.

Lemma (2.4):
Let A be a $\mathbb{Z}/2$ -graded algebra**.** We always have

$$H_0(A, A) = A / [A, A]_{gr} ,$$

where $[A, A]_{gr}$ is the subspace spanned by all graded commutators in A .

Proof. Hochschild complex turns out to be

$$\cdots \longrightarrow C_1(A) \xrightarrow{b} C_0(A) \longrightarrow 0 \qquad \text{i.e.} \cdots \longrightarrow A \otimes A \xrightarrow{b} A \longrightarrow 0 .$$

$H_0(A, A) = \operatorname{coker}(b)$, where, $b : A \otimes A \to A$ is given by $b = d_0 - d_1$, note that $d_0(a_0 \otimes a_1) = a_0 a_1$ and $d_1(a_0 \otimes a_1) = (-1)^{|a_1||a_0|} a_1 a_0$, thus,
$b(a_0 \otimes a_1) = a_0 a_1 - (-1)^{|a_1||a_0|} a_1 a_0 = [a_0, a_1]_{gr}$.$\operatorname{im}(b) = [A, A]_{gr}$.

Since $H_0(A,A)=\mathrm{coker}(b)=A/\mathrm{im}(b)$ then $H_0(A,A)=A/[A,A]_{gr}$.

The map (differential) $b=b_n : C_n(A)\to C_{n-1}(A)$ is given by

$$b(a_0\otimes\ldots\ldots\otimes a_n)=\sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_i a_{i+1}\otimes\ldots\otimes a_n$$
$$+(-1)^{n+|a_n|(|a_0|+\ldots\ldots\ldots|a_{n-1}|)}a_n a_0\otimes a_1\otimes\ldots\ldots\otimes a_{n-1},$$

for example, $b(a_0\otimes a_1)=a_0a_1+(-1)^{1+|a_1||a_0|}a_1a_0=a_0a_1-(-1)^{|a_1||a_0|}a_1a_0=[a_0,a_1]_{gr}$.

A cyclic module defines a map B. Hence we have a mixed complex $(C(A),b,B)$,i.e. a ℤ / 2 -graded complex $(C(A),b)$ with degree $+1$ map B such that $B^2=Bb+bB=0$.

Let us recall how one defines cyclic homology from a mixed complex (M,b,B). It is the homology $HC_*(M)$ of a complex $B(M)$ with $B(M)_n=M_n\oplus M_{n-2}\oplus M_{n-4}\oplus\cdots$. and with differential

$\nabla(m_n\otimes m_{n-2}\otimes m_{n-4}\otimes\ldots)=(bm_n+Bm_{n-2})\otimes(bm_{n-2}+bm_{n-4})\otimes\ldots$.

Such a complex comes with a natural projection $S: B(M)_n\to B(M)_{n-2}$, defined by

$S(m_n\otimes m_{n-2}\otimes m_{n-4},\ldots)=m_{n-2}\otimes m_{n-4}\otimes\ldots,$

which gives $B(M)$ a structure of $HC_*(K)$ -comodule and gives rise to a natural short exact sequence of Z/2-graded complexes, $0\longrightarrow M_n\xrightarrow{I}B(M)_n\xrightarrow{S}B(M)_{n-2}\longrightarrow 0$.

The cyclic homology groups $HC_*(A)$ of a ℤ / 2 -graded algebra A as the cyclic homology groups of the mixed complex $(C(A),b,B)$ defined above,

$HC_*(A)=HC_*(C(A))=H_*(B(C(A)),\nabla)$,

where $B(C(A))$ is defined by

$$B(C(A))_n=(C(A))_n\oplus(C(A))_{n-2}\oplus(C(A))_{n-4}\oplus\ldots$$
$$=C_n(A)\oplus C_{n-2}(A)\oplus C_{n-4}(A)\oplus\ldots\ldots,$$

with differential ∇, given by

$\nabla(a_n\otimes a_{n-2}\otimes a_{n-4}\otimes\ldots)=(b(a_n)+B(a_{n-2}))\otimes(b(a_{n-2})+B(a_{n-4}))\otimes\ldots$.

We have a short exact sequence of ℤ / 2 -graded complexes :

$0\longrightarrow(C(A))_n\xrightarrow{I}B(C(A))_n\xrightarrow{S}B(C(A))_{n-2}\longrightarrow 0$.

A Connes-type long exact sequence between Hochschild and cyclic groups is written as

$$\cdots\longrightarrow HH_n(A)\xrightarrow{I}HC_n(A)\xrightarrow{S}$$
$$\cdots HC_{n-2}(A)\xrightarrow{B}HH_{n-1}(A)\xrightarrow{I}HC_n(A)\xrightarrow{S}\cdots$$

Lemma (2.5):

Let A be a ℤ / 2 -graded algebra , we get

$HC_0(A)=A/[A,A]_{gr}$.

Proof. For small n $(n=0,1)$, we have $\cdots\longrightarrow HH_0(A)\xrightarrow{I}HC_0(A)\xrightarrow{S}\cdots$, and

$\cdots\longrightarrow HH_1(A)\xrightarrow{I}HC_1(A)\longrightarrow 0\longrightarrow HH_0(A)\xrightarrow{I}HC_0(A)\xrightarrow{S}\cdots$,

i.e.

$HC_0(A)\cong HH_0(A)=A/[A,A]_{gr}$

where $HH_n(A)$ for $H_n(A,A)$, for all $n=0,1,2,\ldots$.

We recall the definition and properties of Hochschild and cyclic cohomology groups for ℤ / 2 - graded algebras by dualizing the above complexes from [7].

Let $V = V^{+} \oplus V^{-}$ be a $\mathbb{Z}/2$ -graded vector space .The dual space $V^{*} = Hom_K(V, K)$ can be given a $\mathbb{Z}/2$ -grading by $(V^{*})^{\alpha} = (V^{\alpha})^{*}$, $\alpha = \pm 1$. The associated involution $\theta^{*} : V^{*} \to V^{*}$ is the transpose map of the involution $\theta : V \to V$. We can now dualize the $\mathbb{Z}/2$ -graded complex $(C_*(A), b)$. Denote by $C^{n}(A, A^{*})$ the dual space of $C_n(A)$.

Observe that

$$C^{n}(A, A^{*}) = Hom_K(A^{\otimes n}, A^{*}) = Hom_K(A^{\otimes n}, Hom_K(A, K)) = Hom_K(A^{\otimes n+1}, K) .$$

The transpose of differential b on $C_*(A)$ is a degree $+1$ differential b . If $f \in C^{n-1}(A, A^{*})$ is a n -linear form on A ,then $bf \in C^{n}(A, A^{*})$ will be the $(n+1)$ - linear form.

The differential $b : C^{n-1}(A, A^{*}) \to C^{n}(A, A^{*})$ is given by:

$$bf(a_0 \otimes \ldots\ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^{i} f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) + (-1)^{n+|a_n|(|a_0|+\ldots\ldots|a_{n-1}|)} f(a_n a_0 \otimes a_1 \otimes \ldots. \otimes a_{n-1}) .$$

We note that $bf(a_0 \otimes \ldots. \otimes a_n) = f(b(a_0 \otimes \ldots. \otimes a_n))$, i.e. $bf = fb$.

By definition we get :

$H^{*}(A, A^{*}) = H^{*}(C^{*}(A, A^{*}), b)$.

Lemma (2.6) [7] :

$H^{0}(A, A^{*})$ is the space of all graded traces, i.e. $H^{0}(A, A^{*}) = \{ \text{ all } f \in Hom_K(A, K) \mid f = 0$ on $[A, A]_{gr} \}$, which is the dual space of $H_0(A, A) = A / [A, A]_{gr}$.

Proof. Consider the Hochschild complex :

$0 \longrightarrow Hom_K(A, K) \xrightarrow{b_1} Hom_K(A \otimes A, K) \longrightarrow \cdots$.

We know that $H^{0}(A, A^{*}) = \ker(b_1)$, where;

$b_1 f(a_0 \otimes a_1) = f(a_0 a_1) + (-1)^{1+|a_1||a_0|} f(a_1 a_0) = f(a_0 a_1 - (-1)^{|a_1||a_0|} a_1 a_0) = f([a_0, a_1]_{gr})$.

A cocycle $f \in Hom_K(A, K) = A^{*}$ is inside $\ker(b_1)$ if $0 = b_1 f(a_0 \otimes a_1) = f([a_0, a_1])$, i.e. f vanishes on the subgroup $[A, A]_{gr}$, i.e. $f : A \to K$ is a graded trace. Then

$$H^{0}(A, A^{*}) = \ker(b_1) = \{ \text{ all } f \in Hom_K(A, K) \mid f = 0 \text{ on } [A, A]_{gr} \}$$

Remark :

Since $f : A \to K$ is a graded trace, then $[A, A]_{gr} = 0$, so , $H_0(A, A) = A$ and $Hom_K(H_0(A, A), K) = Hom_K(A, K) = A^{*} = H^{0}(A, A^{*})$.

More generally, Since K is a field, it is clear that $H^{n}(A, A^{*})$ is the dual space of $H_n(A, A)$. Hence $H^{*}(A, A^{*}) = H_*(A, A)^{*}$.

Example (2.7):

For a unital $\mathbb{Z}/2$ -graded algebra. Consider the Hochschild complex :

$$0 \longrightarrow Hom_K(A, K) \xrightarrow{b_1} Hom_K(A \otimes A, K) \ldots \xrightarrow{b_2} Hom_K(A \otimes A \otimes A, K) \longrightarrow \ldots .$$

We know that $H^1(A, A^*) = \dfrac{\ker(b_2)}{im\,(b_1)}$, and $b_1 f\,(a_0 \otimes a_1) = f\,([a_0, a_1]_{gr})$.

The coboundaries in degree 1 are maps $f : A \otimes A \to K$, defined by $f\,(a_0 \otimes a_1) \mapsto f\,([a_0, a_1]_{gr})$ or $a_0 \otimes a_1 \mapsto [a_0, a_1]_{gr} = a_0 a_1 - (-1)^{|a_1||a_0|} a_1 a_0 \in K$.

These functions are K -graded derivations which are called graded inner derivations. Then $im\,(b_1) = \{$ all $f : A \otimes A \to K : a_0 \otimes a_1 \mapsto [a_0, a_1]_{gr}\ \} = \{$Graded inner derivations$\}$,

$b_2 f\,(a_0 \otimes a_1 \otimes a_2) = f\,(a_0 a_1 \otimes a_2) - f\,(a_0 \otimes a_1 a_2) + (-1)^{|a_2|(|a_0|+|a_1|)} f\,(a_2 a_0 \otimes a_1)$,

since f is a cocycle in degree 1, then $b_2 f\,(a_0 \otimes a_1 \otimes a_2) = 0$, i.e.

$$f\,(a_0 \otimes a_1 a_2) = f\,(a_0 a_1 \otimes a_2) + (-1)^{|a_2|(|a_0|+|a_1|)} f\,(a_2 a_0 \otimes a_1). \qquad (2.1)$$

Hence, $f : A \otimes A \to K$ is a graded derivation .Thus we have

$$\ker(b_2) = \{\ \text{all } f \in Hom_K(A \otimes A, K) : f \text{ is a graded derivation}\ \}$$

$$= Der_{gr}(A \otimes A, K) = Der_{gr}(A, A^*),$$

where $Der_{gr}(A, A^*) = Der_{gr}(A, Hom_K(A, K)) = Der_{gr}(A \otimes A, K)$, thus we get $H^1(A, A^*) = Der_{gr}(A, A^*) / \{$Graded inner derivations$\}$.

We recall the definition of cyclic cohomology of a $\mathbb{Z}/2$ -graded algebra. Connes shows that $C^*(A, A^*)$ has an endomorphism B of degree -1 such that $B^2 = Bb + bB = 0$, where A is ungraded algebra. B is the transpose of the degree $+1$ endomorphism B of cyclic homology.

For a graded algebra, we define the $\mathbb{Z}/2$ -graded map $B : C^n(A, A^*) \to C^{n-1}(A, A^*)$ as the transpose of the map $B : C_n(A) \to C_{n+1}(A)$.

We define now the cyclic cohomology of a unital $\mathbb{Z}/2$ -graded algebra A. Denote by $C^*(A, A^*)$ the dual space of $C_*(A)$ .We call $B^n(C^*(A, A^*))$ is the dual complex of the complex $B(C(A)_n)$.

We define the cyclic cohomology groups $HC^*(A)$ of a $\mathbb{Z}/2$ -graded algebra A as

$HC^*(A) = HC^*(C(A)) = H^*(B^*(C^*(A, A^*)), \Delta)$,

where $B^*(C^*(A, A^*))$ is defined by

$$B^n(C^*(A.A^*)) = C^n(A, A^*) \oplus C^{n-2}(A, A^*) \oplus C^{n-4}(A, A^*) \oplus \dots,$$

with differential Δ, given by

$\Delta(c^n \otimes c^{n-2} \otimes c^{n-4} \otimes \dots) = b(c^n) \otimes (B(c^n) + b(c^{n-2})) \otimes (B(c^{n-2}) + b(c^{n-4})) \otimes \dots$.

We have a short exact sequence of $\mathbb{Z}/2$ -graded complexes :

$0 \longrightarrow B^{n-2}(C^*(A, A^*)) \xrightarrow{S} B^n(C^*(A, A^*)) \xrightarrow{I} C^n(A, A^*) \longrightarrow 0$.

A Connes-type long exact sequence between Hochschild and cyclic groups is written as

$$\dots \longrightarrow HC^{n-2}(A) \xrightarrow{S} HC^n(A) \xrightarrow{I}$$

$$\dots H^n(A, A^*) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} \dots$$

Notice also that cyclic cohomology is dual to cyclic homology $HC^n(A) = HC_n(A)^*$.

Definition (2.8):

For a given algebra A. A map $* : A \to A$; $a \mapsto a^*$, is an involution if $*^2 = id_A : A \to A$. For all $a, b \in A$, we have $*(a) = b$ and $*(b) = a$, then $*^2(a) = *(*(a)) = *(b) = a$, i.e. $*^2 = id_A$.

Consider a $\mathbb{Z}/2$ -graded algebra $A = A^+ \oplus A^-$. A graded involution $\theta : A \to A$, is defined by

$$\theta(a) = \alpha a = \begin{cases} +a & if \quad \alpha = + \quad (a \in A^{+}) \\ -a & if \quad \alpha = - \quad (a \in A^{-}) \end{cases},$$

such that $a \mapsto \alpha a \mapsto a$, $\alpha = \pm$, i.e. $\theta^2 = id_A$.

Remark:

The involution given on $C_n(A) = A^{\otimes(n+1)}$, $n = 0,1,..$, by the grading over □ / 2 is given by $\# : C_n(A) \to C_n(A)$, such that $\#(a_0 \otimes ..... \otimes a_n) = \#(a_0) \otimes ..... \otimes \#(a_n)$, for all $a_0,....,a_n \in A$. It commutes with the differential $b_n : C_n(A) \to C_{n-1}(A)$, defined by

$$b_n(a_0 \otimes ...... \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n$$
$$+(-1)^{n+|a_n|(|a_0|+........|a_{n-1}|)} a_n a_0 \otimes a_1 \otimes ... \otimes a_{n-1},$$

and cyclic operator $t_n : C_n(A) \to C_n(A)$, defined by

$t_n(a_0 \otimes ... \otimes a_{n-1} \otimes a_n) = (-1)^{|a_n|(|a_0|+........|a_{n-1}|)} a_n \otimes a_0 \otimes .... \otimes a_{n-1}$.

In other words, $\# b = b \#$ and $\# t = t \#$.

Now, we can define the reflexive operator r .

Definition (2.9):

Let $A = A^+ \oplus A^-$ be a □ / 2 -graded K – algebra with a graded involution

$\# : A \to A \quad ; a \mapsto \alpha a \quad , \alpha = \pm$, for all $a \in A$.

The reflexive operator r acting on $C_n(A) = A^{\otimes(n+1)}, n = 0,1,..$,

by the graded involution $\#$ is given by : $r : C_n(A) \to C_n(A)$ such that;

$$r(a_0 \otimes a_1 \otimes .... \otimes a_n) = \alpha(-1)^{\lambda(\lambda+1)/2} a_0^{\#} \otimes a_n^{\#} \otimes ... \otimes a_1^{\#}, \qquad (2.3)$$

where $\alpha = \pm 1$, $a_i^{\#} = im(a_i)$ under the involution $\#$ and $\lambda = |a_0^{\#}| \sum_{i=1}^{n} |a_i^{\#}| = |a_0^{\#}|(|a_n^{\#}| + ... + |a_1^{\#}|)$. Since $|a_i^{\#}| = |\alpha a_i| = \alpha |a_i|, 0 \le i \le n$, then

$$\lambda = |a_0| \sum_{i=1}^{n} |a_i| = |a_0|(|a_n| + ... + |a_1|). \qquad (2.4).$$

Special cases :

(a) When $\alpha = +$,i.e. $a \in A = A^+$, we have $|a_i| = 0$ and $\lambda = 0$, in (2.3) we have

$r(a_0 \otimes a_1 \otimes ... \otimes a_n) = a_0^{\#} \otimes a_n^{\#} \otimes .... \otimes a_1^{\#}$.

Since $a_i^{\#} = \#(a_i) = a_i$,when $\alpha = +$, then we have

$r(a_0 \otimes a_1 \otimes ..... \otimes a_n) = a_0 \otimes a_n \otimes ..... \otimes a_1$.

(b) When $\alpha = -$, i.e. $a \in A = A^-$, we have $|a_i| = 1$ and $\lambda = |a_0| \sum_{i=1}^{n} 1 = n$, that is $\lambda(\lambda+1)/2 = n(n+1)/2$, in (2.3) we have

$r(a_0 \otimes a_1 \otimes ... \otimes a_n) = -(-1)^{n(n+1)/2} a_0^{\#} \otimes a_n^{\#} \otimes ... \otimes a_1^{\#}$.

Also, $a_i^{\#} = \#(a_i) = -a_i$, when $\alpha = -$, then we have

$r(a_0 \otimes a_1 \otimes ... \otimes a_n) = (-1)^{n(n+1)/2} a_0 \otimes a_n \otimes ... \otimes a_1$.

For example, $r(a_0 \otimes a_1) = \alpha(-1)^{\lambda(\lambda+1)/2} a_0^{\#} \otimes a_1^{\#}$, where $\lambda = |a_0||a_1|, \alpha = \pm$.

III. DIHEDRAL (CO)HOMOLOGY OF $\mathbb{Z}/2$ -GRADED ALGEBRAS.

We introduce the concepts and constructions of dihedral and Reflexive (co)homology and cohomology of $\mathbb{Z}/2$ -graded algebras**.** We use the references [2], [3], and [5].

Let A be a unital $\mathbb{Z}/2$ -graded algebra over K with a graded involution $\theta : A \to A$; $a \mapsto \alpha a$, $\alpha = \pm$.

Consider a dihedral $K[\mathbb{Z}/2]$ -module $C(A) = (C_n(A), b_n, t_n, r_n)$, where $C_n(A) = A^{\otimes(n+1)}, n \geq 0$, and $C_0(A) = A$, $n = 0$, where $b_n : C_n(A) \to C_{n-1}(A)$, given by

$$b_n(a_0 \otimes \ldots\ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$$
$$+(-1)^{n+|a_n|(|a_0|+\ldots\ldots|a_{n-1}|)} a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1},$$

cyclic operator, $t_n : C_n(A) \to C_n(A)$, given by

$$t_n(a_0 \otimes \ldots \otimes a_{n-1} \otimes a_n) = (-1)^{|a_n|(|a_0|+\ldots\ldots|a_{n-1}|)} a_n \otimes a_0 \otimes \ldots \otimes a_{n-1},$$

and reflexive operator , $r_n : C_n(A) \to C_n(A)$, given by

$$r_n(a_0 \otimes a_1 \otimes \ldots. \otimes a_n) = \alpha(-1)^{\lambda(\lambda+1)/2} a_0^{\#} \otimes a_n^{\#} \otimes \ldots \otimes a_1^{\#},$$

where $\alpha = \pm 1$, $a_i^{\#} = im(a_i)$ under the involution θ and $\lambda = |a_0|(|a_n| + \ldots.. + |a_1|)$.

Now, let $A = A^+ \oplus A^-$ be a $\mathbb{Z}/2$ -graded algebra over K .

The algebraic dual space of A , $A^* = Hom(A, K)$ will be given over the $\mathbb{Z}/2$ -grading by $(A^*)^{\alpha} = (A^{\alpha})^*$, $\alpha = \pm$. The associated involution on A^* is the transpose map of involution on A , that is $\# : A \to A ; a \mapsto \alpha a \Rightarrow \#^* : A^* \to A^* ; (a)^* \mapsto (\alpha a)^* = \alpha a^*$.

Let A be a unital $\mathbb{Z}/2$ -graded algebra over K with a graded involution $\theta : A \to A$; $a \mapsto \alpha a$, $\alpha = \pm$.

Consider a codihedral $K[\mathbb{Z}/2]$ -module $C(A) = (C^n(A), b^n, t^n, r^n)$, where $C^n(A) = Hom_K(A^{\otimes n+1}, K), n \geq 0$, is the $\mathbb{Z}/2$ **-**graded space of $(n+1)$ -graded linear maps from A to K . These maps are called cochain maps $f : A^{\otimes n+1} \to K$, $C^0(A) = Hom_K(A, K) = A^*$, where $b^n : C^n(A) \to C^{n+1}(A)$, given by

$$b^n f(a_0 \otimes \ldots\ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n)$$
$$+(-1)^{n+|a_n|(|a_0|+\ldots\ldots|a_{n-1}|)} f(a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}),$$

cyclic operator $t^n : C^n(A) \to C^n(A)$, is

$$t^n f(a_0 \otimes \ldots \otimes a_{n-1} \otimes a_n) = (-1)^{|a_n|(|a_0|+\ldots\ldots|a_{n-1}|)} f(a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}),$$

and reflexive operator $r^n : C^n(A) \to C^n(A)$, given by

$$r^n f(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \alpha(-1)^{\lambda(\lambda+1)/2} f(a_0^{\#} \otimes a_n^{\#} \otimes \ldots \otimes a_1^{\#}),$$

where $\alpha = \pm 1$, $a_i^{\#} = im(a_i)$ under the involution θ and $\lambda = |a_0|(|a_n| + \ldots.. + |a_1|)$.

We expect that :

$$HR_0(A) \cong HC_0(A) \cong HD_0(A)$$
$$\cong A/[A, A]_{gr},$$

for any unital $\mathbb{Z}/2$ -graded algebra A over K where $[A, A]_{gr}$ is the subspace spanned by all graded commutators $[a, b]_{gr} = ab - (-1)^{|a||b|} ba$ in A , and

$$HR^0(A) \cong HC^0(A) \cong HD^0(A)$$
$$\cong H^0(A, A^*),$$

the space of all bounded graded traces, which is the dual space of $H_0(A, A) = A/[A, A]_{gr}$.

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