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A Common Fixed Point Theorem on Fuzzy Metric Space Using Weakly Compatible and Semi-Compatible Mappings

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ABSTRACT: *The aim of this paper is to prove a fixed point theorem in a complete fuzzy metric space using six self maps. We prove our theorem with the concept of weakly compatible mappings and semi-compatible mappings in complete fuzzy metric space.*
KEYWORDS: *Fixed point, self maps, complete fuzzy metric space, semi-compatible mappings, weakly compatible mappings.*
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I. INTRODUCTION AND PRELIMINARIES

Introduction: The concept of Fuzzy sets was introduced by Zadeh[14]. Following the concept of fuzzy sets, fuzzy metric space initiated by Kramosil and Michalek.George and veeramani[5] modified the notion of fuzzy metric spaces with the help of continuous-t norm. Recently, many others proved fixed points theorems involving weaker forms of the compatible mappings in fuzzy metric space.Jungck and Rhoads[3] defined the concept of weakly compatible mappings.Also B.Singh and S.Jain[2] introduced the notion of semi-compatible mappings in fuzzy metric space.

Definition 1.1[1]: A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is called continuous t-norm if $*$ satisfies the following conditions:
(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a*1=a$ for all $a\in[0,1]$
(iv) $a*b \le c*d$ whenever $a \le c$ and $b\le d$ for all $a, b,c,d \in[0,1]$

Definition 1.2[1]: A 3-tuple $(X, M,*)$ is said to be fuzzy metric space if X is an arbitrary set,* is continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$, $s,t>0$
(FM-1) $M(x,y,0)=0$
(FM-2) $M(x,y,t)=1$ for all $t>0$ if and only if $x=y$
(FM-3) $M(x,y,t)= M(y,x,t)$
(FM-4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$
(FM-5) $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous
(FM-6) $\lim_{t\to\infty} M(x,y,t) = 1$

Example 1.3 (Induced fuzzy metric space**)**[1]**:** Let (X, d) be a metric space defined $a*b=\min\{a,b\}$ for all $x,y\in X$ and $t>0$,

$$M(x,y,t) = \frac{t}{t+d(x,y)} \qquad ----(a)$$

Then (X, M,*) is a fuzzy metric space. We call this fuzzy metric *M* induced by metric *d* is the standard fuzzy metric. From the above example every metric induces a fuzzy metric but there exist no metric on X satisfying (a).

Definition 1.4 [1]: Let (X, M,*) be a fuzzy metric space then a sequence $<x_n>$ in X is said to be convergent to $a\ point\ x\in X\ if\ \lim_{n\to\infty} M(x_n,x,t)=1\ \ for\ all\ t>0.$

Definition1.5 **[**1]: A sequence $<x_n>$ in X is called a Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p},x_n,t)=1\ \ for\ all\ t>0\ and\ p>0.$

Definition 1.6 [1]: A fuzzy metric space (X, M,*) is said to be complete if every Cauchy sequence is convergent to a point in X.
Lemma 1.7 [6] : For all $x,y\in X$, M(x,y,.) is non decreasing.
Lemma 1.8 [11] : Let (X,M,*)be a fuzzy metric space if there exists $k\in(0,1)$ such that $M(x,y,kt)\geq M(x,y,t)$ then x=y.
Proposition 1.9 [11] : In the fuzzy metric space (X, M,*) if $a*a\geq a$ for all $a\in[0,1]$ then $a*b=\min\{a,b\}$
Definition 1.10 [12]**:** Two self maps S and T of a fuzzy metric space (X,M,*) are said to be compatible mappings if $\lim_{n\to\infty} M(STx_n,TSx_n,t)=1$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n= z$ for some $z\in X$.
Definition 1.11[2]: Two self maps S and T of a fuzzy metric space (X,M,*) are said to be semi-compatible mappings if $\lim_{n\to\infty} M(STx_n,Tz,t)=1$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n= z$ for some $z\in X$.
Definition 1.12 [3]: Two self maps S and T of a fuzzy metric space (X,M,*) are said to be weakly compatible if they commute at their coincidence point. i.e if Su=Tu for some $u\in X$ then STu=TSu.

II. MAIN RESULT

2.1 Theorem: Let A, B, P,Q ,S and T be self maps of a complete fuzzy metric space (X, M , *) satisfying the conditions,
2.1.1 $AP(X)\subseteq T(X)$ and $BQ(X)\subseteq S(X)$.
2.1.2 The pairs (AP, S) is semi-compatible and (BQ,T) are weakly compatible.
2.1.3 $[M(APx,BQy,kt)]^2 * [M(APx,BQy,kt)M(Ty,Sx,kt)]$

$$\geq \left\{\begin{matrix} k_1[M(BQy,Sx,2kt)*M(APx,Ty,2kt)] \\ +k_2[M(APx,Sx,kt)*M(BQy,Ty,kt]\end{matrix}\right\} M(Ty,Sx,t)$$

for all x, y in X where $k_1,k_2\geq 0, k_1+k_2\geq 1$

2.1.4 AP is continuous mapping.
then A,P, B,Q, S and T have a unique common fixed point in X.
Now we prove a Lemma
2.1.5 Lemma: Let A,P,B,Q, S and T be self mappings from a complete fuzzy metric space (X,M,*) into itself satisfying the conditions 2.1.1 and 2.1.3 then the sequence $\{y_n\}$ defined by $y_{2n}=APx_{2n}=Tx_{2n+1}$ and $y_{2n+1}=BQx_{2n+1}=Sx_{2n+2}$ for $n\geq 0$ relative to four self maps is a Cauchy sequence in X.
Proof: From the conditions 2.1.1 and 2.1.3 and from the definition of iterative sequence we have let x_0 be arbitrary point of X,$AP(X)\subseteq T(X)$ and $BQ(X)\subseteq S(X)$ there exists $x_1,x_2\in X$ such that $APx_0=Tx_1$ and $BQx_1=Sx_2$.
Inductively construct a sequence $\langle x_n\rangle$ and $\langle y_n\rangle$ in X such that $y_{2n}=APx_{2n}=Tx_{2n+1}$ and $y_{2n+1}=BQx_{2n+1}=Sx_{2n+2}$ for $n\geq 0$.
By taking $x=x_{2n}$, $y=x_{2n+1}$ in the inequality 2.1.3 then we get,

$$[M(APx_{2n},BQx_{2n+1},kt)]^2 * [M(APx_{2n,}BQx_{2n+1,}kt)M(Tx_{2n+1},Sx_{2n},kt)]$$

$$\geq\left\{\begin{matrix} k_1[M(BQx_{2n+1},Sx_{2n},2kt)*M(APx_{2n},Tx_{2n+1},2kt]+ \\ k_2[M(APx_{2n},Sx_{2n},kt)*M(BQx_{2n+1},Tx_{2n+1},kt)]\end{matrix}\right\} M(Tx_{2n+1},Sx_{2n},t)$$

$$[M(y_{2n},y_{2n+1},kt)]^2 * [M(y_{2n,}y_{2n+1,}kt)M(y_{2n},y_{2n-1},kt)]$$

$$\geq\left\{\begin{matrix} k_1[M(y_{2n+1},y_{2n-1},2kt)*M(y_{2n},y_{2n},2kt]+ \\ k_2[M(y_{2n},y_{2n-1},kt)*M(y_{2n+1},y_{2n},kt)]\end{matrix}\right\} M(y_{2n},y_{2n-1},t)$$

this implies

$$[M(y_{2n},y_{2n+1,},kt)]\{M(y_{2n},y_{2n+1},kt)*M(y_{2n},y_{2n-1},kt)\}$$

$$\geq\left\{\begin{matrix} k_1[M(y_{2n+1},y_{2n-1},2kt)]+ \\ k_2[M(y_{2n},y_{2n-1},kt)*M(y_{2n+1},y_{2n},kt)]\end{matrix}\right\} M(y_{2n},y_{2n-1},t) \text{ and}$$

$$[M(y_{2n},y_{2n+1},kt)]\{M(y_{2n+1},y_{2n-1},2kt\}$$

$$\geq\{k_1[M(y_{2n+1},y_{2n-1},2kt]+k_2[M(y_{2n-1},y_{2n+1},2kt)]\}M(y_{2n},y_{2n-1},t)$$

$$[M(y_{2n},y_{2n+1},kt)][M(y_{2n+1},y_{2n-1},2kt)]$$

$$\geq[k_1+k_2][M(y_{2n+1},y_{2n-1},2kt][M(y_{2n},y_{2n-1},t]$$

This gives
$M(y_{2n},y_{2n+1},kt)\geq(k_1+k_2)M(y_{2n-1},y_{2n},t)$
Since $k_1+k_2\geq 1$
$M(y_{2n},y_{2n+1},kt)\geq M(y_{2n-1},y_{2n},t)$
this implies $M(y_n,y_{n+1},kt)\geq M(y_{n-1},y_n,t)$

Continuing in this process we get

$$M(y_n, y_{n+1,} t) \geq M\left(y_{n-1}, y_n, \frac{t}{k}\right) \geq M\left(y_{n-1}, y_n, \frac{t}{k^2}\right) \geq M\left(y_{n-1}, y_n, \frac{t}{k^3}\right) \ldots$$
$$\ldots . \geq M\left(y_{n-1}, y_n, \frac{t}{k^n}\right)$$

And this implies $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$.
Now for each ε>0 and t>0 we can choose $n_0 \in N$ such that
$M(y_n, y_{n+1}, t) > 1 - \varepsilon$ for m, n ∈N. Suppose m ≥ n

$$M(y_n, y_m, t) \geq \begin{bmatrix} M\left(y_n, y_{n+1}, \frac{t}{m-n}\right) * M\left(y_{n+1}, y_{n+2,} \frac{t}{m-n}\right) * \ldots \\ * M\left(y_{m-1}, y_m, \frac{t}{m-n}\right) \end{bmatrix}$$
$$\geq (1-\varepsilon)*(1-\varepsilon)*\ldots\ldots(1-\varepsilon)$$
$$\geq (1-\varepsilon)$$

this shows that the sequence $\{y_n\}$ is a Cauchy sequence in complete fuzzy metric space X and hence it converges to a limit, say z∈X

Proof of theorem 2.1:

From the Lemma $APx_{2n} \to z$, $Tx_{2n+1} \to z$, $BQx_{2n+1} \to z, Sx_{2n+2} \to z$ as n→∞
Since AP is continuous $AP(Sx_{2n+2}) \to z$ and $AP(APSx_{2n}) \to APz$ as n→∞
Also since the pair (AP,S) is semi-compatible implies $AP(Sx_{2n+2}) \to Sz$ as n→∞ therefore APz=Sz.
Put x= z, y=x_{2n+1} in the inequality 2.1.3 then, we get

$$[M(APz, BQx_{2n+1}, kt)]^2 * [M(APz, BQx_{2n+1}, kt)M(Tx_{2n+1}, Sz, kt)]$$
$$\geq \begin{Bmatrix} k_1[M(BQx_{2n+1}, Sz, 2kt) * M(APz, Tx_{2n+1}, 2kt)] \\ +k_2[M(APz, Sz, kt) * M(BQx_{2n+1}, Tx_{2n+1}, kt)] \end{Bmatrix} M(Tx_{2n+1}, Sz, t)$$
$$[M(APz, z, kt)]^2 * [M(APz, z, kt)M(z, APz, kt)]$$
$$\geq \begin{Bmatrix} k_1[M(z, APz, 2kt) * M(APz, z, 2kt)] \\ +k_2[M(APz, APz, kt) * M(z, z, kt)] \end{Bmatrix} M(z, APz, t)$$
$$[M(APz, z, kt)]^2 \geq \begin{Bmatrix} k_1[M(z, APz, 2kt)] \\ +k_2[1] \end{Bmatrix} M(z, APz, t)$$
$$[M(APz, z, kt)] \geq k_1 M(APz, z, kt) + k_2$$
$$(1 - k_1)M(APz, z, kt) \geq k_2$$
$$M(APz, z, kt) \geq \frac{k_2}{(1 - k_1)} \geq 1$$

Implies APz=Sz=z
Also from the condition AP(X)⊆T(X) , there exists w∈X such that APz=Tw=z.
Now to prove BQw=z
Put x= z , y=w in the inequality 2.1.3 then, we get

$$[M(APz, BQw, kt)]^2 * [M(APz, BQw, kt)M(Tw, Sz, kt)]$$
$$\geq \begin{Bmatrix} k_1[M(BQw, Sz, 2kt) * M(APz, Tw, 2kt)] \\ +k_2[M(APz, Sz, kt) * M(BQw, Tw, kt)] \end{Bmatrix} M(Tw, Sz, t)$$
$$[M(z, BQw, kt)]^2 * [M(z, BQw, kt)M(z, z, kt)]$$
$$\geq \begin{Bmatrix} k_1[M(BQw, z, 2kt) * M(z, z, 2kt)] \\ +k_2[M(z, z, kt) * M(BQw, z, kt)] \end{Bmatrix} M(z, z, t)$$

this implies

$$[M(z, BQw, kt)]^2 \geq \begin{Bmatrix} k_1[M(BQw, z, 2kt)] \\ +k_2[M(BQw, z, kt)] \end{Bmatrix}$$
$$[M(z, BQw, kt)]^2 \geq \begin{Bmatrix} k_1[M(BQw, z, kt)] \\ +k_2[M(BQw, z, kt)] \end{Bmatrix}$$

$M(z, BQw, kt) \geq k_1 + k_1 \geq 1$ implies
BQw=z
Hence BQw=Tw=z.
Now the pair (BQ,T) is weakly compatible implies BQ(Tw)=T(BQ)w and this gives BQz=Tz.
Put x=z,y=z in the inequality 2.1.3 then, we get

$$[M(APz, BQz, kt)]^2 * [M(APz, BQz, kt)M(Tz, Sz, kt)]$$
$$\geq \begin{Bmatrix} k_1[M(BQz, Sz, 2kt) * M(APz, Tz, 2kt)] \\ +k_2[M(APz, Sz, kt) * M(BQz, Tz, kt)] \end{Bmatrix} M(Tz, Sz, t)$$
$$[M(z, BQz, kt)]^2 * [M(z, BQw, kt)M(BQz, z, kt)]$$

$$\geq \begin{Bmatrix} k_1[M(BQz,z,2kt) * M(z,BQ,2kt)] \\ +k_2[M(z,z,kt) * M(BQz,BQz,kt)] \end{Bmatrix} M(BQz,z,t)$$

this implies

$$[M(z,BQz,kt)]^2 \geq \begin{Bmatrix} k_1[M(BQz,z,kt)] \\ +k_2[1] \end{Bmatrix} M(BQz,z,t)$$

$$M(z,BQz,kt)(1-k_1) \geq k_2$$

$$M(z,BQz,kt) \geq \frac{k_2}{(1-k_1)} \geq 1$$

Implies BQz=z.

Put x=Pz and y=z in the inequality 2.1.3 then, we get

$$[M(AP(Pz),BQz,kt)]^2 * [M(AP(Pz),BQz,kt)M(Tz\,,S(Pz),kt)]$$
$$\geq \begin{Bmatrix} k_1[M(BQz,S(Pz),2kt) * M(AP(Pz),Tz,2kt)] \\ +k_2[M(AP(Pz),S(Pz),kt) * M(BQz,Tz,kt)] \end{Bmatrix} M(Tz,S(Pz),t)$$

$$[M(Pz,z,kt)]^2 * [M(Pz,z,kt)M(z\,,Pz,kt)]$$
$$\geq \begin{Bmatrix} k_1[M(z,Pz,2kt) * M(Pz,z,2kt)] \\ +k_2[M(Pz,Pz,kt) * M(z,z,kt)] \end{Bmatrix} M(z,Pz,t)$$

this implies

$$[M(Pz,z,kt)]^2 \geq \begin{Bmatrix} k_1[M(Pz,z,kt)] \\ +k_2[1] \end{Bmatrix} M(z,Pz,t)$$

$$M(Pz,z,kt)(1-k_1) \geq k_2$$

$M(Pz,z,kt) \geq \frac{k_2}{(1-k_1)} \geq 1$ implies Pz=z then

APz=z gives A[P(z)]=z consequently Az=z.

Put x=z and y=Qz in the inequality 2.1.3 then, we get

Put x=z,y=Qz in the inequality 2.1.3 then, we get

$$[M(APz,BQ(Qz),kt)]^2 * [M(APz,BQ(Qz),kt)M(T(Qz)\,,Sz,kt)]$$
$$\geq \begin{Bmatrix} k_1[M(BQ(Qz),Sz,2kt) * M(APz,T(Qz),2kt)] \\ +k_2[M(APz,Sz,kt) * M(BQ(Qz),T(Qz),kt)] \end{Bmatrix} M(T(Qz),Sz,t)$$

$$[M(z,Qz,kt)]^2 * [M(z,Qz,kt)M(Qz\,,z,kt)]$$
$$\geq \begin{Bmatrix} k_1[M(Qz,z,2kt) * M(z,Qz,2kt)] \\ +k_2[M(z,z,kt) * M(Qz,Qz,kt)] \end{Bmatrix} M(Qz,z,t)$$

this implies

$$[M(z,Qz,kt)]^2 \geq \begin{Bmatrix} k_1[M(Qz,z,kt)] \\ +k_2[1] \end{Bmatrix} M(Qz,z,t)$$

$$M(Qz,z,kt)(1-k_1) \geq k_2$$

$M(Qz,z,kt) \geq \frac{k_2}{(1-k_1)} \geq 1$ this implies Qz=z then

BQz=z gives B[Q(z)]=z consequently Bz=z.

Therefore Az=Bz=Qz=Pz=Tz=Sz=z.

Uniqueness:Let $z^*(\neq z)$ be the another fixed point of the mappings A,B,P,Q,S and T then Az*=Bz*=Pz*=Qz*=Sz*=Tz*=z*.

Put x=z and y=z* in the inequality 2.1.3 then, we get

Put x=z,y=z*

$$[M(APz,BQz^*,kt)]^2 * [M(APz,BQz^*,kt)M(Tz^*\,,Sz,kt)]$$
$$\geq \begin{Bmatrix} k_1[M(BQz^*,Sz,2kt) * M(APz,Tz^*,2kt)] \\ +k_2[M(APz,Sz,kt) * M(BQz^*,Tz^*,kt)] \end{Bmatrix} M(Tz^*,Sz,t)$$

$$[M(z,z^*,kt)]^2 * [M(z,z^*,kt)M(z^*\,,z,kt)]$$
$$\geq \begin{Bmatrix} k_1[M(z^*,z,2kt) * M(z,z^*,2kt)] \\ +k_2[M(z,z,kt) * M(z^*,z^*,kt)] \end{Bmatrix} M(z^*,z,t)$$

this implies

$$[M(z,z^*,kt)]^2 \geq \begin{Bmatrix} k_1[M(z^*,z,kt)] \\ +k_2[1] \end{Bmatrix} M(z^*,z,t)$$

$$M(z,z^*,kt) \geq \frac{k_2}{(1-k_1)} \geq 1$$

Implies z*=z.

Hence the self mappings A,B,S,T,P, and Q has a unique common fixed point.

REFERENCES

[1] A George , P Veeramani , On some results in fuzzy metric spaces, *Fuzzy Sets and Systems, 64*, 1994, 395-399.
[2] B Singh , S Jain ,Semi –compatible and fixed point theorems in fuzzy metric space , *Chungcheong Math. Soc. 18, 2005, 1-22.*
[3] G Jungck , and B E Rhoads , Fixed point for set valued function without continuity ,*Indian J.Pure and Appl.Math.29(3)*,1998,227-238
[4] G Jungck , Compatible mappings and common fixed points , Internet.*J.Math & Math. Sci.* , 9(4), 1986, 771-779.
[5] Ivan Kramosil and Michalek, "Fuzzy Metrics and Statistical Metric Spaces" *Kybernetika,* 11, 1975,336-334
[6] M Grabiec , Fixed points in fuzzy metric spaces, *Fuzzy Sets and Systems, 27*, 1988, 385-389.
[7] P.Balasubramaniam ,S. Murali Sankar S & R.P Pant , Common fixed points of four mappings in fuzzy metric space, *J.Fuzzy Math. 10(2),* 2002, 379-384
[8] R.P.Pant & K. Jha , A remark on, Common fixed points of four mappings in a fuzzy metric space, J.*Fuzzy Math.12 (2)*,2004,433-437.
[9] S Kutukcu , S Sharma & H A Tokgoz ,A fixed point theorem in fuzzy metric spaces, *Int. J. Math. Analysis*, 1(18), 2007 , 861-872.
[10] S N Mishra ,N Sharma &S L Singh , Common fixed points of mappings on fuzzy metric spaces, *Internet .J.Math &Math .Sci*,17,1994, 253-258.
[11] S.Sharma , Common fixed point theorems in fuzzy metric spaces, *Fuzzy Sets and Systems*, *125*,2001, 1-8.
[12] Y J Cho, H K Pathak , S M Kang &J S Jung , Common fixed point of compatible maps of type β on fuzzy metric spaces, *Fuzzy sets and Systems, 93*,1998, 99-111.
[13] V Srinivas,B.V.B.Reddy,R.Umamaheswarrao,A Common fixed point Theorem on Fuzzy metric Spaces,*Kathmandu University journal of Science , Engineering and Technology,8 (II),*2012,77-82.
[14] L .A Zadeh L, Fuzzy Sets*, Information and Control,8,1*965,338-353