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Teaching Mathematics Concepts via Computer Algebra Systems

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ABSTRACT: *Most articles examine computer algebra systems (CAS) as they relate to the teaching and learning of mathematics from advantages to disadvantages. This paper will explore junior undergraduate students' ability to solve distinguish tricky examples using various CAS technologies. Additionally, an understanding for how CAS technologies are adopted and applied in professional environments is valuable, both in guiding improvements to these tools and identifying new tools which can aid mathematicians.*
KEYWORDS: *Computer Algebra Systems (CAS), Mathematics Concepts, Limits, Integration, Differential Equations*

I. INTRODUCTION

Since the need for tools in mathematical problem solving, many computer science researchers have created tools to support mathematical problem solving. One way to enhance mathematical problem solving is to use computer algebra system (CAS) technology. CAS is a program that manipulates both symbols and numbers to reduce the algebraic manipulation of algebra; in the same way calculators (i.e., scientific and four-function) can reduce computation for time and work involved in arithmetic. Some CAS technology combine a wide range of mathematical functions with graphing capabilities. As such they have the potential to reduce the time students spend calculating algorithms, allowing more time for concept development, which isenhanced by the ability of the program to concurrently represent mathematical function graphically. The point of view of the user will vary depending on the application and the intended outcome. Such as, a statistician may have different considerations than a physicist, which may in turn be different from that of a mathematician, and/or a mathematics educator. Each type of user has certain goals and desires for CAS technology and how this technology will aid in his or her quest to achieve a certain solution to a problem.

II. METHODS AND FUTURE CONTENTS OF MATHEMATICS COURSES

We argue that it is important to let junior undergraduate students, especially those whose major is not STEM related, solve distinguish tricky mathematics problems first with CAS. We define distinguish tricky mathematics problem to be a problem where a CAS solution is not accurate and the student would need to know more about the mathematics concept and CAS technology to make a clear determination for the solution. In other words, the students must recognize that he/she must manipulate the CAS in order to arrive to a correct solution. By allowing students with non-STEM related majors opportunities to solve distinguish tricky mathematics problems first with CAS, allows the student time to explore the concepts, rather than, the tedious mathematical manipulations to solve the problem. Additionally, such students typically have questions regarding the use of CAS, which can help drive instruction and lead students to a greater understanding of the mathematical concept.
For the purpose of this article, four different CAS technologies were used. The four CAS technologies are: Texas Instruments, Matlab, Mathematica, and Microsoft Mathematics Calculator. Matlab, Mathematica, and Microsoft Mathematics Calculator are all computer-based CAS technologies. [1] (Rashwan, 2014) gives some examples solved by Matlab and Mathematica. However, Texas Instruments (TI) is the only hand-held CAS technology explored (TI-Nspire CX CAS). As noted by [2] (Johnson, 2010), the TI-Nspire CX CAS has "advanced features as, symbolic manipulation, constructing geometric representations, and exploring multiple representations (i.e., algebraically and graphically) dynamically all on one screen" (p. 44). In addition, Microsoft Mathematics Calculator is the only free CAS technology, while the others require a one-time fee.
Again, the mission of this article is to explore the idea of solving distinguish tricky mathematics problems using CAS. Further investigating the need to interrupt the solution with more advanced mathematics skills.

III. LIMIT AND CONTINUITY OF ONE VARIABLE FUNCTION

Distinguish Tricky Problem 1: Is
$$f(x)=\begin{cases}\dfrac{x+3x^2+5x^3+\cdots+(2n-1)x^n-n^2}{x-1}, & x\neq 1\\[2ex] \dfrac{n(n^2-1)}{3}, & x=1\end{cases}$$
continuous?

Texas Instrument:	Matlab:

$\sum_{k=1}^{n}((2\cdot k-1)\cdot x^k)$
 $\frac{x^{n+1}\cdot((2\cdot n-1)\cdot x-2\cdot n-1)}{(x-1)^2}+\frac{x\cdot(x+1)}{(x-1)^2}$
 $\lim_{x\to 1}\left(\frac{x^{n+1}\cdot((2\cdot n-1)\cdot x-2\cdot n-1)}{(x-1)^2}+\frac{x\cdot(x+1)}{(x-1)^2}\right)$
 n^2
	`>> symsum((2*k-1)*x^k, k, 1, n)`
	 `ans =`
	 `x^(n+1)*(-3*x-1+2*(n+1)*x-2*n)/(x-1)^2-x*(-x-1)/(x-1)^2`
	 `limit(((x^(n+1)*(-3*x-1+2*(n+1)*x-2*n)/(x-1)^2-x*(-x-1)/(x-1)^2)-n^2)/(x-1),x,1)`
	 $- 1/6\ n + 1/2\ n^2 + 2/3\ n^3$

Mathematica: $\sum_{k=1}^{n}(2k-1)\,x^k$ $\frac{x\,(1+x-x^n-2\,n\,x^n-x^{1+n}+2\,n\,x^{1+n})}{(-1+x)^2}$ $\text{Limit}\left[\left(\frac{x\,(1+x-x^n-2\,n\,x^n-x^{1+n}+2\,n\,x^{1+n})}{(-1+x)^3}-\frac{n^2}{(-1+x)}\right),\ x\to 1\right]$ $\frac{1}{6}n\,(-1+3\,n+4\,n^2)$	Microsoft Mathematics Calculator: Cannot help solving this problem

Remark 1: The main point here is to find the sum of $x + 3x^2 + 5x^3 + \cdots + (2n-1)x^n$, and then find $\lim_{x\to 1}\left[\frac{x(1+x-x^n-2nx^n-x^{n+1}+2nx^{n+1})}{(x-1)^3}-\frac{n^2}{x-1}\right]$.

The function is not continuous when the value of the limit is not equal to the value of the function at $x = 1$. The limit involves L'Hopial's rule but we cannot do it before having the sum of $x + 3x^2 + 5x^3 + \cdots + (2n-1)x^n$.

IV. DEFINITE INTEGRAL

Distinguish Tricky Problem 2: $\int_{-1}^{1}\frac{1}{x}$

Texas Instrument:	Matlab:

$\int_{-1}^{1}\frac{1}{x}\,dx$ undef	`>> syms x;int(1/x,-1,1)`
	 `ans =`
	 `NaN`

Mathematica:	Microsoft Mathematics Calculator:

In[1]:= $\int_{-1}^{1}\frac{1}{x}\,dx$ Integrate::idiv : Integral of $\frac{1}{x}$ does not converge on {−1, 1}. » Out[1]= $\int_{-1}^{1}\frac{1}{x}\,dx$	Input $\int_{-1}^{1}\frac{1}{x}\,dx$ Output $\int_{-1}^{1}\frac{1}{x}\,dx$

Remark 2: The function $f(x)$ must be continuous on [a,b] to find $\int_a^b f(x)dx$ and its value is the area under the curve $f(x)$ on the interval [a,b]. The function $f(x)=\frac{1}{x}$ is not continuous at $x=0\in[-1,1]$. By ignoring the continuity condition, the result is $\int_{-1}^{1}\frac{1}{x}dx=\ln(1)-\ln(-1)=0-\ln(-1)=-\ln(-1)$ which is undefined value.

Distinguish Tricky Problem 3: $\int_{-1}^{1}\frac{1}{x^2}dx$

Texas Instrument:
$\int_{-1}^{1}\frac{1}{x^2}\,dx$ ∞
	Matlab:
	`>> syms x;int(1/x^2,-1,1)`
	`ans =`
	`Inf`

Mathematica: In[1]:= `Integrate[1/x^2, {x, -1, 1}]` Integrate::idiv : Integral of $\frac{1}{x^2}$ does not converge on {−1, 1}. » Out[1]= $\int_{-1}^{1}\frac{1}{x^2}\,dx$	Microsoft Mathematics Calculator: 5 Input $\int_{-1}^{1}\frac{1}{x^2}\,dx$ Output $\int_{-1}^{1}\frac{1}{x^2}\,dx$

Remark 3: Completing this without technology would give $\left[\frac{-1}{x}\right]_{-1}^{1}=-1-1=-2$ and this is an incorrect answer.

Improper Integral

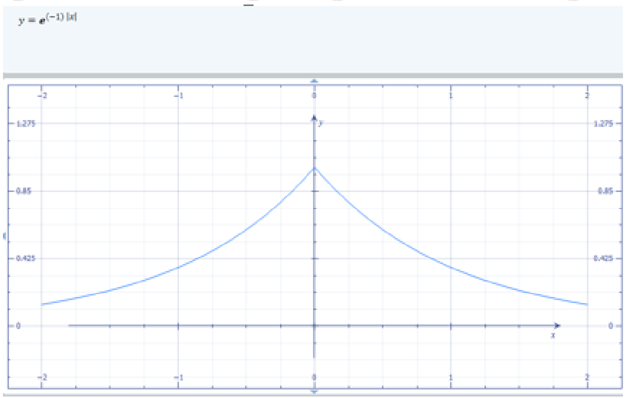
Distinguish Tricky Problem 4: The improper integral $\int_0^{\infty}e^{-|x|}dx$.

Texas Instrument:
$\int_0^{\infty}e^{-\lvert x\rvert}\,dx$ 1
	Matlab:
	`>> syms x; int(exp(-abs(x)),x,0,inf)`
	`ans =`
	`1`

Mathematica:	Microsoft Mathematics Calculator:
1 Input $\int_0^{\infty} e^{-\lvert x\rvert}\,dx$ Output $\int_0^{\infty} \frac{1}{e^{\lvert x\rvert}}\,dx$	In[3]:= $\int_0^{\infty} e^{-\mathrm{Abs}[x]}\,dx$ Out[3]= $\begin{cases} \frac{1}{\mathrm{Log}[e]} & \mathrm{Re}[\mathrm{Log}[e]] > 0 \\ \mathrm{Integrate}[e^{-\mathrm{Abs}[x]}, \{x, 0, \infty\}, \mathrm{Assumptions} \to \mathrm{Re}[\mathrm{Log}[e]] \le 0] & \mathrm{True} \end{cases}$

Remark 4: Let us graph $e^{-\lvert x\rvert}$ by the Matlab command,

```
>> x=-10:1:10;y=-10:1:10;y=exp(-abs(x));plot(x,y)
```



So, $e^{-\lvert x\rvert} = e^{-x}, 0 \le x < \infty$, then it is easy to complete without technology

$$\int_0^{\infty} e^{-\lvert x\rvert}dx = \int_0^{\infty} e^{-x}dx = \left[-e^{-x} \right]_0^{\infty} = 0 - (-1) = 1 .$$

Imaginary Integration

Distinguish Tricky Problem 5: $\int_{-2}^{1} \ln\left(\frac{x-1}{x+2}\right)dx$

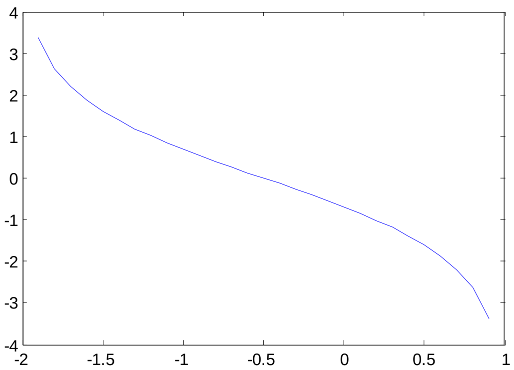
Texas Instrument:	Matlab:

$\int_{-2}^{1} \ln\left(\frac{x-1}{x+2}\right) dx$ $3\cdot\pi\cdot i$	`>> syms x;int(log((x-1)/(x+2)),x,-2,1)`
	`ans =`
	`3*i*pi`

Mathematica:	**Microsoft Mathematics Calculator:**
In[4]:= `Integrate[Log[(x - 1) / (x + 2)], {x, -2, 1}]` Out[4]= $3\, i\, \pi$	Input $\int_{-2}^{1} \ln\left(\frac{x-1}{x+2}\right) dx$ Output $\int_{-2}^{1} \ln\left(\frac{x-1}{x+2}\right) dx$

Remark 5: To graph of $f(x) = \ln\left(\frac{x-1}{x+2}\right), -2 \le x \le 1$ by Matlab:

```
>> x=-2:0.1:1;plot(x,log((x-1)./(x+2)))
Warning: Imaginary parts of complex X and/or Y arguments ignored
```



Based on the output warning provided by Matlab, the solution refers to ignoring the imaginary parts of $f(x)$ when $x = -2$ and $x = 1$.

Integration Test

Distinguish Tricky Problem 6: To check the convergent of the series $\sum_{n=3}^{\infty} \frac{\ln(n)}{\ln(\ln(n))}$ by the integration test, is extremely difficult without technology. It is recommended to use CAS to make sense of the example.

Texas Instrument: $\int_3^{\infty} \frac{\ln(x)}{\ln(\ln(x))}\,dx$ ∞	Matlab: `>> syms n;int(log(n)/log(log(n)),3,Inf)` `ans =` `Inf`
Mathematica: In[2]:= $\int_3^{\infty} \frac{\text{Log}[x]}{\text{Log}[\text{Log}[x]]}\,dx$ Integrate::idiv : Integral of $\frac{\text{Log}[x]}{\text{Log}[\text{Log}[x]]}$ does not converge on $\{3, \infty\}$. ≫ Out[2]= $\int_3^{\infty} \frac{\text{Log}[x]}{\text{Log}[\text{Log}[x]]}\,dx$	Microsoft Mathematics Calculator: Input $\int_3^{\infty} \frac{\ln(x)}{\ln(\ln(x))}\,dx$ Output $\int_3^{\infty} \log_{\ln(x)} x\,dx$

Remark 6: (I) The result is "ans = Inf " means " ∞ " which means the series is divergent. But the series is convergent when the integration value is a finite value.
(II) The concept involves the definite integral which could be discussed and explored by CAS. For instance, the area under a curve in an interval in its domain.

V. DOUBLE INTEGRAL AND POLAR COORDINATES

$\iint_R f(x, y)\,dA$ is the volume of the solid region between a bounded rectangular or non- rectangular region R in the plane R and the surface $Z = f(x, y)$. It is the extension to the definite integral but for a continuous function of two variables $f(x, y)$.

Distinguish Tricky Problem 7: $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dxdy$ and $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dydx$.

Texas Instrument:
$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy$ 1.34954
	Matlab:
	`>> syms x y;int(int(exp(x^2+y^2),x,0,sqrt(1-y^2)),y,0,1)`
	`Warning: Explicit integral could not be found.`
	`> In sym.int at 58`
	`ans =`
	`int(-1/2*i*erf(i*(1-y^2)^(1/2))*exp(y^2)*pi^(1/2),y = 0 .. 1)`

Mathematica: In[2]:= $\int_0^1 \int_0^{\sqrt{1-y^{\wedge}2}} \mathrm{Exp}[x^{\wedge}2+y^{\wedge}2]\,dx\,dy$ Out[2]= $\int_0^1 \frac{1}{2} e^{y^2} \sqrt{\pi}\ \mathrm{Erfi}\left[\sqrt{1-y^2}\right] dy$	Microsoft Mathematics Calculator: Input $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$ Output $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$

Remark 7: Some of CAS tried to compute the integration of two variable function in the Cartesian coordinates, the importance of transforming into the polar coordinate and the area element in it $dA = r dr d\theta = dxdy$ comes to account. The bound of the integration for r and θ are found from the relation between the Cartesian and the polar coordinates. None of the CAS technologies provided a correct solution. For instance, transforming into polar coordinates by CAS, by using Mathematica the output is:

In[8]:= **fp = Simplify[With[{x = r Cos[θ], y = r Sin[θ]}, f = $e^{x^2+y^2}$]]**

Out[8]= e^{r^2}

Then integration is possible using $\int_0^{\frac{\pi}{2}} \int_0^1 e^{r^2} r dr d\theta = \frac{(e-1)\pi}{4}$, that can be done by non-CAS technology. However, Texas Instrument provided the same result as $\frac{(e-1)\pi}{4} = 1.34954$.

Reversing the integration order

Distinguish Tricky Problem 8: $\int_0^{\frac{1}{16}} \int_{\sqrt[4]{y}}^{\frac{1}{2}} \cos(16\pi x^5)\,dxdy$

Texas Instrument:
$\int_0^{\frac{1}{16.}} \int_{\sqrt[4]{y}}^{\frac{1}{2}} \cos(16\cdot\pi\cdot x^5)\,dx\,dy$ 0.003979
	Matlab:
	`>> syms x y;int(int(cos(16*pi*x^5),x,y^(1/4),0.5),y,0,1/16)`
	`Warning: Explicit integral could not be found.`

Mathematica: In[13]:= `f[x_, y_] := Cos[16*Pi*x^5]` $\int_0^{\frac{1}{16}} 1 \left(\int_{\sqrt[4]{y}}^{\frac{1}{2}} f[x, y]\,dx\right) dy$ Out[14]= $\frac{1}{80\pi}$	Microsoft Mathematics Calculator: Input $\int_0^{\frac{1}{16}} \int_{\sqrt[4]{y}}^{\frac{1}{2}} \cos(16\pi x^5)\,dx\,dy$ Output $\int_0^{\frac{1}{16}} \int_{\sqrt[4]{y}}^{\frac{1}{2}} \cos(16\pi x^5)\,dx\,dy$

Remark 8: SomeCAS technology cannot evaluate some double integrationwithout reversing it and can help after doing so. Next the importance of reversing the integration is shown:

Texas Instrument:	Matlab:

$\int_0^{\frac{1}{16.}} \int_{\sqrt[4]{y}}^{\frac{1}{2}} \cos(16\cdot\pi\cdot x^5)\,dx\,dy$ 0.003979	`>> syms x y;int(int(cos(16*pi*x^5),y,0,x^4),x,0,0.5)`
	`ans =`
	`1/80/pi`

Mathematica:	Microsoft Mathematics Calculator:
In[13]:= `f[x_, y_] := Cos[16*Pi*x^5]` $\int_0^{\frac{1}{16}} \left(\int_{\sqrt[4]{y}}^{\frac{1}{2}} f[x, y]\,dx\right) dy$ Out[14]= $\frac{1}{80\,\pi}$	Input $\int_0^{\frac{1}{2}} \int_0^{x^4} \cos(16\,\pi\,x^5)\,dy\,dx$ Output $\frac{1}{80\,\pi}$ Decimal Output 0.0039788735773

VI. DIFFERENTIAL EQUATION

Distinguish Tricky Problem 9: The following first order differential equation:

$$\frac{dy}{dx} = 2y(x\sqrt{y} - 1),\ y(0) = 1$$

Texas Instrument:	Matlab:

$deSolve(y' = 2\cdot y\cdot(x\cdot\sqrt{y} - 1) and\, y(0) = 1, x, y)$
$\frac{1}{\sqrt{y}} = 1 + x$
	`>> dsolve('Dy = 2*y*(x*sqrt(y)-1)','y(1)=0','x')`
	`ans =`
	`0`

Mathematica:	Microsoft Mathematics Calculator:
In[1]:= `DSolve[{y'[x] == 2 y[x] (x Sqrt[y[x]] - 1), y[0] == 1}, y[x], x` Out[1]= $\left\{\left\{y[x] \to \frac{1}{(-1 + 2e^x - x)^2}\right\}, \left\{y[x] \to \frac{1}{(1+x)^2}\right\}\right\}$	Cannot solve differential equations

Remark 9: (I) the two solution are $y = \frac{1}{(1 - 2e^x + x)^2}$, and $y = \frac{1}{(1 + x)^2}$ satisfying the differential equation and its initial condition.

(II) Without the initial condition Matlab gives:

```
                y = 0
1/y^(1/2)-1-x-exp(x)*C1 = 0
```

Where, C1 = 0 or C1 =-2 by using the initial condition $y(0) = 1$, therefore the solution is coincide with

VII. CONCLUSION

1. CAS is not only used for solving time-consuming mathematics problems but also is used to explore mathematics concepts by solving distinguish tricky problems.
2. Sometimes CAS will not help students in solving some mathematics problems and the students must be aware when to use and not use CAS.
3. Use of CAS features to focus on mathematics concepts and how to adopt the solution technique avoiding the theoretical calculation steps.

4. Mathematical background and mathematical software skills are necessary in solving mathematical problems when using CAS technology.
5. Mathematics course descriptions should include three parts. First, teaching of mathematics should link to the students' major. Second, CAS should be an essential component in mathematical problem solving. Third, some mathematical problems should relate to the student's major.

6. An understanding of how these tools are adopted and applied in professional environments is valuable, both in guiding improvements to these tools and in identifying new tools which can aid mathematicians.

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