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Matrix Transformations on Paranormed Sequence Spaces Related To De La Vallée-Pousin Mean

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ABSTRACT: *In this paper, we determine the necessary and sufficient conditions to characterize the matrices which transform paranormed sequence spaces into the spaces $V_\sigma(\lambda)$ and $V_\sigma^\infty(\lambda)$, where $V_\sigma(\lambda)$ denotes the space of all (σ,λ)-convergent sequences and $V_\sigma^\infty(\lambda)$ denotes the space of all (σ,λ)-bounded sequences defined using the concept of de la Vallée-Pousin mean.*

Keywords: *de la Vallée-Pousin Mean, σ-convergence, Invariant Mean, Matrix Transformations*
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I. INTRODUCTION

We shall denote the space of real valued sequences by ω.Any vector subspace of ω is called a sequence space. if $x \epsilon \omega$, then we write $x = (x_k)$ instead of $x = (x_k)_{k=0}^\infty$. We denote the spaces of all finite, bounded, convergent and null sequences by $, l_\infty, c,$ and c_0, respectively. Further, we shall use the conventions that $e = (1, 1, 1,$ ...and *e(k)* as the sequence whose only non zero term is 1 in the k*th* place for each $k \epsilon N$,

A sequence space X with linear topology is called a K-space if each of the maps $p_i : X \to \mathbb{C}$: defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space is called an FK-space if X is complete linear metric space; a BK-space is a normed FK-space.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a sub additive function $p : X \to \mathbb{R}$ such that $p(\theta) = 0, p(x) = (-x)$, and scalar multiplication is continuous, i. e. $|\alpha_n - \alpha| \to 0$ and $p(x_n) \to 0$ imply $p(\alpha_n x_n - \alpha x) \to 0 \; as \; n \to \infty$ for all $x \in X$ and $\alpha \in \mathbb{R}$, where θ is the zero vector in the linear space X.

Assume here and after that $x = (x_k)$ be a sequence such that $x_k \neq 0, \forall\, k \epsilon \mathbb{N}$ and (p_k) be the bounded sequence of strictly positive real numbers with $supp_k = H \; and \; M = \max(1 , H)$.then , the sequence spaces

$$c_0(p) = \{ x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\}$$
$$c(p) = \{ x = (x_k) \in \omega : \lim_{k\to\infty} |x_k - l|^{p_k} = 0, for \; some \; l \in \mathbb{C}\}$$
$$\ell_\infty(p) = \{ x = (x_k) \in \omega : \sup_k |x_k|^{p_k} < \infty \} \text{ and}$$
$$\ell(p) = \{ x = (x_k) \in \omega : \textstyle\sum_k^\infty |x_k|^{p_k} < \infty \}$$

were defined and studied by Maddox [1] and Simons [2].

If $p_k = p , \forall\, k \in \mathbb{N}$ for some constant $p > 0$, then these spaces reduce to $c_0 , c, \ell_\infty, and \; \ell_p$, respectively. Note that $c_0(p)$ is a linear metric space paranormed by $h(x) = \sup_k |x_k|^{\frac{p_k}{M}}$. $\ell_\infty(p)$, $c(p)$ fail to be linear metric space because the continuity of multiplication does not hold for them. These two spaces turn out to be linear metric spaces if and only if $\inf p_k > 0$. $\ell(p)$ is linear metric space paranormed by $w(x) = (\sum_k |x_k|^{p_k})^{\frac{1}{M}}$.

These sequence spaces are complete paranormed spaces in their respective paranorm if and only if $\inf p_k > 0$. However, these are not normed spaces, in general. (see Aydin and Basar [3] and Karakaya et al, [4]).

The above sequence spaces were further generalized. Bulut and Çakar [5] defined the sequence space

$$\ell(p,s) = \{ x = (x_k) \epsilon \omega : \textstyle\sum_k k^{-s} |x_k|^{p_k} < \infty, s \geq 0\},$$

which generalized the sequence space $\ell(p)$. They showed that $\ell(p,s)$ isa linear sequence space paranormed by

$$g(x) = (\textstyle\sum_k k^{-s}|x_k|^{p_k})^{\frac{1}{M}}.$$

Basarir [6] generalized the other sequence spaces as follows:

$$\ell_\infty(p,s) = \{ x = (x_k) \in \omega : \sup_k k^{-s} |x_k|^{p_k} < \infty \}$$
$$c_0(p,s) = \{ x = (x_k) \in \omega : k^{-s}|x_k|^{p_k} \to 0, (k \to \infty)\}$$
$$c(p,s) = \{ x = (x_k) \in \omega : k^{-s}|x_k - l|^{p_k} \to 0, for \; some \; l, (k \to \infty)\}$$

It is easy to see that $c_0(p,s)$ is paranormed by $q(x) = \sup_k(k^{-s} |x_k|^{p_k})^{\frac{1}{M}}$. Also $\ell_\infty(p,s)$ and $c(p,s)$ are paranormed by $q(x)$ iff $inf p_k > 0$. All the spaces defined above are complete in their topologies.

Let X and Y be two sequence spaces and $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of real or complex numbers. We denote $Ax = (A_n(x))$, $A_n(x) = \sum_k a_{nk} x_k$ provided that the series on the right converges for each n. If $x = (x_k) \in X$, implies that $Ax \in Y$, then we say that A defines a matrix transformation from X into Y, and by (X, Y) we denote the class of such matrices.

Let σ be a one to one mapping from the set $\mathbb{N}$ of natural numbers into itself. A continuous linear functional φ on the space ℓ_∞ is said to be an invariant mean or σ-mean if and only if
(i) $\varphi(x) \geq 0$ if $x \geq 0, (i.e\ x_k \geq 0$, for all $k)$,
(ii) $\varphi(e) = 1$, where $e = (1,1,1,\ldots)$,
(iii) $\varphi(x) = \varphi\left(\left(x_{\sigma(k)}\right)\right)$, for all $x \in \ell_\infty$.
Though out this we consider the mapping σ which has no finite orbit, that is $\sigma^p(k) \neq k$ for all integer $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ denotes the p-th iterate of σ at k. Note that a σ −mean extends the limit functional on the space c in the sense that $\sigma(x) = limx$ for all $x \in \mathbb{C}$, (see Mursaleen [7]). Consequently, $c \subset V_\sigma$, the set of bounded sequence all of whose σ −means are equal.
We say that a sequence $x = (x_k)$ is σ −convergent iff $x \in V_\sigma$. Using this concept, Schaefer [8] defined and characterized σ −convervative and σ −coercive matrices. If σ is translation, then V_σ is reduced to f of almost convergent sequences (Lorentz [9]). As an application of almost convergence, Mohiuddine [10] established some approximation theorems for sequences of positive linear operators through this concept. The idea of σ −convergence for double sequences was introduced in Çakar et al [11].

II. BASIC DEFINITIONS

Definition 2.1 (de la Vallée-Poussin mean): Let $\lambda = (\lambda_m)$ be a non –decreasing sequence of positive numbers tending to ∞ such that $\lambda_{m+1} \leq \lambda_m + 1,\ \lambda_1 = 0$, then

$$\rho_m(x) = \frac{1}{\lambda_m}\sum_{j\in I_m} x_j$$

is called the generalized de la Vallée-Poussin mean, where $I_m = [m - \lambda_m + 1,\ m\,]$.
Definition 2.2 (Mursaleen et al [12]) A sequence $x = (x_k)$ of real numbers is said to be (σ,λ) −convergent to a number L iff

$$\lim_{m\to\infty}\frac{1}{\lambda_m}\sum_{j\in I_m} x_{\sigma^j(n)} = L$$

uniformly in n, and $V_\sigma(\lambda)$ denotes the set of all such sequences i.e

$$V_\sigma(\lambda) = \{\, x \in \ell_\infty :\ \lim_{m\to\infty}\frac{1}{\lambda_m}\sum_{j\in I_m} x_{\sigma^j(n)} = L, \text{ uniformly in n}\}$$

Note that a convergent sequence is (σ,λ) −convergent but converse need not hold.

2.1 Remark
(i) If $\sigma(n) = n + 1$, then $V_\sigma(\lambda)$ is reduced to the f_λ (see Mursaleen et al [13])
(ii) If $\lambda_m = m$, then $V_\sigma(\lambda)$ is reduced to the space V_σ
(iii) If $\sigma(n) = n + 1$ and $\lambda_m = m$, then $V_\sigma(\lambda)$ is reduced to the space f, (almost convergent sequences)
(iv) $c \subset V_\sigma(\lambda) \subset \ell_\infty$.

Definition 2.3 (Mohiuddine [13]) A sequence $x = (x_k)$ of real numbers is said to be (σ,λ) −bounded if and only if $sup_{m,n}\left|\frac{1}{\lambda_m}\sum_{j\in I_m} x_{\sigma^j(n)}\right| < \infty$, and we denote by $V_\sigma^\infty(\lambda)$, the set of all such sequences i. e

$$V_\sigma^\infty(\lambda) = \{\, x \in \ell_\infty :\ \sup_{m,n}|t_{mn}(x)| < \infty\,\},$$

where,

$$t_{mn}(x) = \frac{1}{\lambda_m}\sum_{j\in I_m} x_{\sigma^j(n)}$$

2.2 Remark
$c \subset V_\sigma(\lambda) \subset V_\sigma^\infty(\lambda) \subset \ell_\infty$.

III. SOME KNOWN RESULTS

The following results play vital role in our main results
Lemma 3.1 (Theorem 1; Bulut and Çakar [5]):
(i) If $1 < p_k \leq \sup_k p_k = H < \infty, and\ {p_k}^{-1} + {q_k}^{-1} = 1$,for k$\in \mathbb{N}$, then

$$\ell^\dagger(p,s) = \{\, a = (a_k): \sum_{k=1}^\infty k^{s(q_k-1)} N^{-\frac{q_k}{p_k}}|a_k|^{q_k} < \infty, s > 0, for\ some\ N > 1\}$$

(ii) If $0 < m = inf_k p_k \leq p_k \leq 1$, for each k =1,2,3, …, then $\ell^\dagger(p,s) = m(p,s)$, where

$$m(p,s) = \{\, a = (a_k): \sup_k k^{-s}|a_k|^{p_k} < \infty, s \geq 1$$

Lemma 3.2 (Theorem3; Bulut and Çakar [5]): (i) If $1 < p_k \leq \sup_k p_k = H < \infty$, for every $k \in \mathbb{N}$, then $A \in (\ell(p,s),\ell_\infty)$ if and only if there exists an integer N> 1, such that

$$\sup_n \sum_{k=1}^\infty |a_{nk}|^{q_k} N^{-q_k} k^{s(q_k-1)} < \infty, \tag{3.1}$$

(ii) If $0 < m = inf_k p_k \leq p_k \leq 1$, for each $k \in \mathbb{N}$, then $A \in (\ell(p,s),\ell_\infty)$ if and only if

$$\sup_{nk}|a_{nk}|^{p_k} k^s = M < \infty. \tag{3.2}$$

Lemma 3.3 (Theorem 4; Bulut and Çakar [5]):
(i) Let $1 < p_k \leq \sup_k p_k = H < \infty$ for every k. Then $A \in (\ell(p,s),c)$ if and only if (3.1) holds together with

$$a_{nk} \to \alpha_k,\ (n \to \infty, k\ (fixed)) \tag{3.3}$$

(ii) If $0 < m = inf_k p_k \leq p_k \leq 1$, for some $k \in \mathbb{N}$.Then $A \in (\ell(p,s),\ c)$, if and only if condition (3.2) and (3.3)

Lemma 3.4 (Theorem 2.1 Mohiuddine [13]): The spaces $V_\sigma(\lambda)$ and $V_\sigma^\infty(\lambda)$ are BK spaces with the norm

$$||x|| = \sup_{m,n\geq 0} |t_{mn}\ (x)|. \tag{3.4}$$

Lemma 3.5 (Theorem 3.1 Mohiuddine [13]): Let $1 < p_k \leq \sup_k p_k = H < \infty$, for every $k \in \mathbb{N}$.Then $A \in (\ell(p),\ V_\sigma^\infty(\lambda))$ if and only if there exists an integer $N > 1$ such that

$$\sup_{m,n} \sum_k |\frac{1}{\lambda_m} \sum_{j\in I_m} a_{\sigma^j(n),k}|^{q_k} N^{-q_k} < \infty, \tag{3.5}$$

Lemma 3.6 (Theorem 3.2 Mohiuddine [13]):

(i) $1 < p_k \leq \sup_k p_k = H < \infty$, for every $k \in \mathbb{N}$.Then $A \in (\ell(p), V_\sigma(\lambda))$ if and only if (i) condition (3.5)

(ii) $\lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{j\in I_m} a_{\sigma^j(n),k} = \alpha_k$ uniformly in n, for every $k \in \mathbb{N}$

IV. MAIN RESULTS

We shall prove the following results.

Theorem 4.1 Let $1 < p_k \leq \sup_k p_k = H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (\ell(p,s), V_\sigma^\infty(\lambda))$ if and only if there exists an integer $D > 1$, such that

$$\sup_{m,n} \sum_k |\frac{1}{\lambda_m} \sum_{j\epsilon I_m} a_{\sigma^j(n),k}|^{q_k} D^{-q_k} k^{s(q_k-1)} < \infty. \tag{4.1}$$

Proof. Sufficiency. Let (4.1) hold and that $x \in \ell(p,s)$.Using the inequality

$|a\, b| \leq D(|a|^q D^{-q} + |b|^p)$ for $D > 0$ and a, b complex numbers $(p^{-1} + q^{-1} = 1)$ (Maddox [14])

We have

$$|t_{mn}\,(Ax)| = \sum_k |\frac{1}{\lambda_m} \sum_{j\in I_m} a_{\sigma^j(n),k}\, x_k|$$

$$\Rightarrow \quad \sum_k D[|\frac{1}{\lambda_m} \sum_{j\epsilon I_m} a_{\sigma^j(n),k}|^{q_k} D^{-q_k}. k^{s(q_k-1)} + k^{-s}|x_k|^{p_k}],$$

where ${p_k}^{-1} + {q_k}^{-1} = 1$.

Taking the supremum over m, n on both sides and using (4.1), we get

$Ax \in V_\sigma^\infty(\lambda)$ for $x \in \ell(p,s)$. i. e $A \in (\ell(p,s), V_\sigma^\infty(\lambda))$.

Necessity. Let $A \in \big(\ell(p,s), V_\sigma^\infty(\lambda)\big)$. We put $q_n(x) = \sup_k k^{-s} |t_{mn}\,(Ax)|$.

It is easy to see that for $n \geq 0, q_n$ is a continuous semi norm on $\ell(p,s)$ and (q_n) is point wise bounded on $\ell(p,s)$. Suppose that (4.1) is not true. Then there exists $x \in \ell(p,s)$ with $\sup_n q_n\,(x) = \infty$. By the principle of condensation of singularities (Yosida [15]), the set $\{\, x \in \ell(p,s){:} \sup_n q_n\,(x) = \infty\}$ is of second category in $\ell(p,s)$ and hence non empty. Thus there exists $x \in \ell(p,s)$ with $\sup_n q_n\,(x) = \infty$.But this contradict the fact that (q_n) is pointwise bounded on $\ell(p,s)$. Now by the Banach-Steinhaus theorem, there is a constant M such that

$$q_n(x) \leq Mg(x) \tag{4.2}$$

Now define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} \delta^{\frac{M}{p_k}}(sgn\frac{1}{\lambda_m}\sum_{j\epsilon I_m} a_{\sigma^j(n),k})|\frac{1}{\lambda_m}\sum_{j\epsilon I_m} a_{\sigma^j(n),k}|^{q_k-1}V^{-1}D^{-\frac{q_k}{p_k}}.k^{s(q_k-1)} & 1 \leq k \leq k_0 \\ 0, & k > k_0 \end{cases}$$

where $0 < \delta < 1$ and $V = \sum_{k=1}^{k_0} |\frac{1}{\lambda_m} \sum_{j\in I_m} a_{\sigma^j(n),k}|^{q_k} D^{-q_k}. k^{s(q_k-1)}$.

Then it is easy to see that $x \in \ell(p,s)$ and $g(x) \leq \delta$.

Applying this sequence to (4.2) we get the condition (4.1)

This completes the proof of the theorem.

Theorem 4.2 Let $1 < p_k \leq \sup_k p_k = H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (\ell(p,s), V_\sigma(\lambda))$ if and only if

(i) Condition (4.1) of Theorem 4.1 holds

(ii) $\lim_m \frac{1}{\lambda_m} \sum_{j\in I_m} a_{\sigma^j(n),k} = \alpha_k$, uniformly in n for every $k \in \mathbb{N}$.

Proof. Sufficiency. Let (i) and (ii) hold and $x \in \ell(p,s)$. For $j \geq 1$

$$\sum_{k=1}^{j} |\frac{1}{\lambda_m} \sum_{j\in I_m} a_{\sigma^j(n),k}|^{q_k} D^{-q_k}. k^{s(q_k-1)} \leq \sup_m \sum_k |\frac{1}{\lambda_m} \sum_{j\in I_m} a_{\sigma^j(n),k}|^{q_k} D^{-q_k}. k^{s(q_k-1)} < \infty$$

for every n. Therefore

$$\sum_k |\alpha_k|^{q_k}\, D^{-q_k}. k^{-s(q_k-1)} = \lim_j \lim_k \sum_{k=1}^{j} |\frac{1}{\lambda_m} \sum_{j\epsilon I_m} a_{\sigma^j(n),k}|^{q_k}. D^{-q_k}. k^{-s(q_k-1)}$$

$$\leq \sup_k \sum_k \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} \right|^{q_k} D^{-q_k} . k^{s(q_k - 1)} < \infty,$$

where $p_k{}^{-1} + q_k{}^{-1} = 1$. Consequently reasoning as in the proof of the sufficiency of Theorem 4.1, the series $\frac{1}{\lambda_m} \sum_k \sum_{j \in I_m} a_{\sigma^j(n),k} x_k$ and $\sum_k \alpha_k x_k$ converge for every n, m, j *and for every* $x \in \ell(p,s)$. For a given $\varepsilon > 0$ *and* $x \in \ell(p,s)$, choose k_0 such that

$$\left(\sum_{k=k_0+1}^{\infty} k^{-s} |x_k|^{p_k}\right)^{\frac{1}{M}} \varepsilon, \tag{4.3}$$

where $M = \sup_k p_k$ condition (ii) implies that there exists m_0 such that

$$\left| \sum_{k=1}^{m_0} \left[\frac{1}{\lambda_m} \sum_{j \epsilon I_m} a_{\sigma^j(n),k} - \alpha_k \right] \right| < \frac{\varepsilon}{2}$$

for all $m \geq m_0$ and uniformly in n. Now, since

$\frac{1}{\lambda_m} \sum_k \sum_{j \in I_m} a_{\sigma^j(n),k} x_k$ and $\sum_k \alpha_k x_k$ converge (absolutely) uniformly in m, n and for every $x \in \ell(p,s)$, we have $\left| \sum_{k=k_0+1}^{\infty} \left[\frac{1}{\lambda_m} \sum_{j \epsilon I_m} a_{\sigma^j(n),k} - \alpha_k \right] x_k \right.$ converges uniformly in m, n. Hence by conditions (i) and (ii)

$$\left| \sum_{k=k_0+1}^{\infty} \left[\frac{1}{\lambda_m} \sum_{j \epsilon I_m} a_{\sigma^j(n),k} - \alpha_k \right] \right| < \frac{\varepsilon}{2}$$

for $m \geq m_0$ and uniformly in n. Therefore

$$\left| \sum_{k=k_0+1}^{\infty} \left[\frac{1}{\lambda_m} \sum_{j \epsilon I_m} a_{\sigma^j(n),k} - \alpha_k \right] x_k \right| \to \infty, (m \to \infty) \text{ , uniformly in } n \text{, i. e}$$

$$\lim m \sum_k \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} x_k = \sum_k \alpha_k x_k \tag{4.4}$$

Necessity. Let $A \in (\ell(p,s), V_\sigma(\lambda))$. Since $V_\sigma(\lambda) \subset V_\sigma^\infty(\lambda)$. Condition (i) follows by Theorem 4.1. Since $e^{(k)} = (0,0,0,\ldots,1(kth \text{ place}),0,0,\ldots) \in \ell(p,s)$ and condition (ii) follows immediately by (4.4).
This completes the proof of the theorem.

V. CONCLUSION

The notion of invariant mean and de la Vallée-Poussin mean plays very active role in the recent research on matrix transformations. With the help of these two notions, the concept of (σ,λ)-convergent sequences denoted by $V_\sigma(\lambda)$ and (σ,λ)-bounded sequences denoted by $V_\sigma^\infty(\lambda)$ and also (σ,θ)-convergent sequence, $(V_\sigma(\theta))$ and (σ,θ)-bounded sequence, $(V_\sigma^\infty(\theta))$ sequences and many others have evolved. Related to these sequence spaces, many matrix classes have been characterized. As we have characterized the matrix classes $(\ell(p,s), V_\sigma(\lambda))$ and $(\ell(p,s), V_\sigma^\infty(\lambda))$ in our main results here, some other characterizations may also evolve.

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