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A Characterization of the Zero-Inflated Logarithmic Series Distribution

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ABSTRACT: *In this paper, we introduce a characterization of the zero-inflated logarithmic series distributions through a linear differential equation of its probability generating function.*
Keywords*: Zero-Inflated Logarithmic Series Distribution, Probability Generating Function, Linear Differential Equation.*

I. INTRODUCTION

Satheesh Kumar and Riyaz (2013) developed an extended version of the zero-inflated logarithmic series distribution (ZILSD) and derive some of its structural and estimating its parameters. Then they (2015a, b ,c)) developed an order k version of the ZILSD and considered some of its structural properties, considered a modified version of logarithmic series distribution and study some of its properties, and proposed an alternative version of the ZILSD and studied some of its properties, and some of its applications.
Logarithmic series distribution (LSD) sometime called logarithmic distribution or log-series distribution. This distribution is a member of the class of generalized power series distributions. For a detailed historical remarks and genesis of the LSD see Johnson et al (2005, 303-305).
In this paper, we introduce in Section 2, the definition of the LSD and ZILSD with their probability generating function (pgf), followed in Section 3 we characterize the ZILSD through a linear equation of its pgf.

II. THE LOGARITHMIC AND THE ZERO-INFLATED LOGARITHMIC SERIES DISTRIBUTIONS

Let $\theta \in (0,1)$, then the discrete random variable (rv) X having probability mass function (pmf);

$$P(X = x) = \begin{cases} \dfrac{\theta^x}{-x\log(1-\theta)} & x = 1,2,3,\ldots \\ 0 & otherwise, \end{cases} \tag{2.1}$$

is said to have a logarithmic distribution (LSD) with parameter θ. We will denote that by writing $X \sim LSD(\theta)$. See Johnson et al (2005), pp 302-325, for further details.
The pgf of the rv X, $G_X(t)$, is given by;

$$\begin{aligned} G_X(t) &= E(t^X) \\ &= \frac{1}{-\log(1-\theta)}\sum_{x=1}^{\infty}\frac{\theta^x}{x}t^x \\ &= \frac{\log(1-\theta t)}{\log(1-\theta)} \end{aligned} \tag{2.2}$$

Let $\alpha \in (0,1)$ be an extra proportion added to the proportion of zero of the rv X, then the rv Y defined by;

$$P(Y = y) = \begin{cases} \alpha, & y = 0 \\ (1-\alpha)\dfrac{\theta^y}{-y\log(1-\theta)} & y = 1,2,3,\ldots \\ 0 & otherwise, \end{cases} \tag{2.3}$$

is said to have a ZILSD, and we will denote that by writing $Y \sim ZILSD(\theta;\alpha)$.
Note that, if $\alpha \to 0$, then (2.3) reduces to the standard form of the LSD given by (2.1).
The pgf of the rv Y, can be shown to be;

$$G_Y(t) = \alpha + (1-\alpha)\frac{\log(1-\theta t)}{\log(1-\theta)} \tag{2.4}$$

III. CHARACTERIZATION OF THE ZERO-INFLATED LOGARITHMICSERIES DISTRIBUTION

We give below the main result of this paper.

Theorem: Let $G(t)$ be the pgf of a discrete rv Y taking non-negative integer values then the rv Y has a degenerate, a Bernoulli or a ZILSD if its pgf satisfies, for some numbers a, b and c, that;

$$(a+bt)\frac{\partial}{\partial t}G(t)=c \tag{3.1}$$

Proof: Let us consider all possible values of the numbers a, b and c.

Case 1: $a=0,\ b=0$ and $c=0$is a non-sense case.

Case 2: $a\neq 0,\ b=0$ and $c=0$. We have that; $a\frac{\partial}{\partial t}G(t)=0$, or equivalently, $\frac{\partial}{\partial t}G(t)=0$, hence, $G(t)=k$, where k is a constant. Since 1 = $G(1)$; we have that k = 1; and therefore $G(t)=1$, inducting that the rv Y is a degenerate at 0.

Case 3: $a=0,\ b\neq 0$ and $c=0$. We have that; $bt\frac{\partial}{\partial t}G(t)=0$, hence; $G(t)=k$, where k is a constant, resulting that the rv Y is a degenerate at 0 also.

Case 4: $a=0,\ b=0$ and $c\neq 0$. We have that; c = 0, a non-sense case.

Case 5: $a=0,\ b\neq 0$ and $c\neq 0$. We have that; $bt\frac{\partial}{\partial t}G(t)=c$, or equivalently;

$$\frac{\partial}{\partial t}G(t)=\frac{c}{bt}$$

Hence, $P(Y=0)=\frac{\partial}{\partial t}G(0)$ is not defined, and therefore this case is not possible.

Case 6: $a\neq 0,\ b=0$ and $c\neq 0$. We have that; $a\frac{\partial}{\partial t}G(t)=c$, or equivalently; $\frac{\partial}{\partial t}G(t)=\frac{c}{a}$; hence the solution is;

$$G(t)=\frac{c}{a}\mathrm{t}+k$$

Where k is a constant. Since 1 = $G(1)$; we have that k = 1; and therefore; $k=1-\frac{c}{a}$, hence;

$$G(t)=1-\frac{c}{a}+\frac{c}{a}\mathrm{t}$$

Now, if a and c satisfy that $0<\frac{c}{a}<1$, or equivalently, $0<|c|<|a|<\infty$; then the rv Z has a Bernoulli distribution with parameter $\theta=\frac{c}{a}$.

Case 7: $a\neq 0,\ b\neq 0$ and $c=0$. We have that; $(a+b)\frac{\partial}{\partial t}G(t)=0$, or equivalently, $\frac{\partial}{\partial t}G(t)=0$, hence; $G(t)=k$, where k is a constant, resulting that the rv Y is a degenerate at 0 also.

Case 8: $a\neq 0,\ b\neq 0$ and $c\neq 0$. Without loss of generality, we can assume $a=1$ in (3.1), hence it becomes;

$$(1+bt)\frac{\partial}{\partial t}G(t)=c \tag{3.2}$$

Or equivalently;

$$\frac{\partial}{\partial t}G(t)=\frac{c}{1+bt}$$

Resulting in that;

$$G(t)=\frac{c}{b}\log(1+bt)+k$$

Where k is a constant, determined from the fact that $G(1)=1$, that is;

$$k=\ 1-\frac{c}{b}\log(1+b)$$

Hence,

$$G(t)=\frac{c}{b}\log(1+bt)+1-\frac{c}{b}\log(1+b) \tag{3.3}$$

Using (3.3), we have that;

$$P(Y=0)=G(0)=1-\frac{c}{b}\log(1+b) \tag{3.4}$$

Now;

$$\frac{\partial}{\partial t}G(t)=\frac{c}{1+bt}$$

Therefore,

$$P(Y=1)=\frac{\partial}{\partial t}G(0)=c \tag{3.5}$$

Let us show that $0<P(Y=1)<1$. Suppose that $P(Y=1)=1,$ then from (3.5), we have that c=1, and therefore, $P(Y=0)=0$, that is; $1-\frac{1}{b}\log(1+b)=0$, or equivalently, $\log(1+b)=b$. But; $\log(1+b)=b$ has solution $b=0$, which is contradict the assumption that $b\neq 0$. Since by assumption $c\neq 0$, we have that $0<P(Y=1)<1$, hence

$$0<c<1 \tag{3.6}$$

Therefore, $0 < P(Y = 0) < 1$, that is

$$0 < 1 - \frac{c}{b}\log(1 + b) < 1 \tag{3.7}$$

It can be seen that;

$$\frac{\partial^{(n)}}{\partial t^n} G(t) = \frac{c(n-1)!\,(-b)^{n-1}}{(1+bt)^n}, \qquad n = 1, 2, 3, \ldots \tag{3.8}$$

It follows that;

$$P(Y =) = \frac{1}{y!}\frac{\partial^{(y)}}{\partial t^y} G(0)$$

$$= \frac{c(-b)^{y-1}}{y}, \qquad y = 1, 2, 3, \ldots \tag{3.9}$$

Using (3.6), we have that;

$$0 \le \frac{(-b)^{y-1}}{y} < 1, \qquad y = 1, 2, 3, \ldots \tag{3.10}$$

Let us consider all possible values of b. Firstly, we note that in order for $G(t)$ given by (3.3) to be a pgf, $b \neq -1$; since it will not be defined when $b = -1$, therefore $b \neq -1$. Secondly, b cannot be positive, since if it does, then the quantity $\frac{(-b)^{y-1}}{y}$ will be negative for even positive integer number of y. Similarly, if $b < -1$, then the quantity $\frac{(-b)^{y-1}}{y}$ will be unbounded as y getting large and hence violated (3.9). Therefore, $-1 < b < 0$.

Writing

$$\alpha = 1 - \frac{c}{b}\log(1 + b) \tag{3.11}$$

Hence,

$$c = (1 - \alpha)\frac{-\theta}{log(1 - \theta)}$$

Where $\theta = -b$, and note that $0 < \theta < 1$ and that $0 < \alpha < 1$ from (3.7), and hence the rv Y ~ $ZILSD(\theta; \alpha)$ with pmf given by (2.3).

Theorem 2: Let Z be a discrete rv taking non-negative integer values, then Z ~ $ZILSD(\theta; \alpha)$, for some non-zero θ and αif and only if its pgf satisfying (3.1) for some non-zero numbers a, b and c.

Proof: let Z ~ $ZILSD(\theta; \alpha)$, for some θ *and* α, then its pgf is given by (2.4), hence;

$$\frac{\partial}{\partial t} G(t) = (1 - \alpha)\frac{1}{log(1 - \theta)}\frac{-\theta}{(1 - \theta t)}$$

And hence,

$$(a + b)\frac{\partial}{\partial t} G(t) = c$$

With $a = 1$, $b = -\theta$ and $c = (1 - \alpha)\frac{-\theta}{log(1-\theta)}$, therefore, (3.1) is satisfied, and hence the proof is complete using Case 7 of Theorem 1.

IV. CONCLUSIONS

We introduced a characterization of the zero-inflated logarithmic series distributions through a linear differential equation of its probability generating function. We would propose an extension of this results to other forms as well as to others distributions.

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