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On Some Multiple Integral Formulas Involving Jacobi and Laguerre Polynomials of Several Variables

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ABSTRACT: *The aim of the present paper is to derive some multiple integral formulas involving Jacobi and Laguerre polynomials of several variables. These results are established with the help of a known and interesting integrals given in Edwards [2].Furthermore, some special cases are also derived.*

KEYWORDS: *multiple integral formulas, Jacobi polynomials, Laguerre polynomials Kamp´e de F´eriet function*

I. INTRODUCTION

The Jacobi polynomials of several variables $P_n^{(\alpha_1,\beta_1;\cdots;\alpha_r,\beta_r)}(x_1,\cdots,x_r)$ of Shrivastava [7]are defined as follows :

$$P_n^{(\alpha_1,\beta_1;\cdots;\alpha_r,\beta_r)}(x_1,\cdots,x_r)=\frac{(1+\alpha_1)_n(1+\alpha_2)_n\cdots(1+\alpha_r)_n}{(n!)^r}$$

$$\times F^{1:1;\cdots;1}_{0:1;\cdots;1}\left[\begin{matrix}-n:1+\alpha_1+\beta_1+n\ ;\cdots;1+\alpha_r+\beta_r+n\ ; \\ -:\quad \alpha_1+1\quad ;\cdots;\quad \alpha_r+1\quad ;\end{matrix}\ \frac{1-x_1}{2},\cdots,\frac{1-x_r}{2}\right] \tag{1.1}$$

where $F^{p:q_1;\cdots;q_n}_{l:m_1;\cdots;m_n}[x_1,\cdots,x_n]$ is the multivariable extension of the Kamp´e de F´eriet function [8], see also [9]

$$F^{p:q_1;\cdots;q_n}_{l:m_1;\cdots;m_n}[z_1,\cdots,z_n]\equiv F^{p:q_1;\cdots;q_n}_{l:m_1;\cdots;m_n}\left(\begin{matrix}(a_p):(b'_{q_1})\ ;\cdots\ ;(b^{(n)}_{q_n})\ ; \\ (c_l):(d'_{m_1})\ ;\cdots\ ;(d^{(n)}_{m_n})\ ;\end{matrix}\ z_1,\cdots,z_n\right)$$

$$=\sum_{s_1,\cdots s_n=0}^{\infty}\Omega(s_1,\cdots,s_n)\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!}, \tag{1.2}$$

where

$$\Omega(s_1,\cdots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\cdots+s_n}\prod_{j=1}^{q_1}(b'_j)_{s_1}\cdots\prod_{j=1}^{q_n}(b^{(n)}_j)_{s_n}}{\prod_{j=1}^{l}(c_j)_{s_1+\cdots+s_n}\prod_{j=1}^{m_1}(d'_j)_{s_1}\cdots\prod_{j=1}^{m_n}(d^{(n)}_j)_{s_n}}\ . \tag{1.3}$$

And for convergence of the multivariable hypergeometric series in (1.3)

$1+l+m_k-p-q_k\geq 0, k=1,\cdots,n;$

The equality holds when, in addition, either

$p>l$ *and* $|x_1|^{\frac{1}{p-l}}+\cdots+|x_n|^{\frac{1}{p-l}}<1;$

or

$p\leq l$ *and* $\max\{|x_1|,\cdots,|x_n|\}<1$.

The Laguerre polynomials of several variables $L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ of khan and Shukla [3] are defined as follows:

$$L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)=\frac{(1+\alpha_1)_n\cdots(1+\alpha_r)_n}{(n!)^r}\Psi_2^{(r)}\left[-n;\alpha_1+1,\cdots,\alpha_r+1;x_1,\cdots,x_r\right] \quad (1.4)$$

where $\Psi_2^{(r)}$ is the confluent hypergeometric function of r-variables [9]

$$\Psi_2^{(r)}\left[a;c_1,\cdots,c_r;x_1,\cdots,x_r\right]=\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{(a)_{m_1+\cdots+m_r}}{(c_1)_{m_1}\cdots(c_r)_{m_r}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \quad (1.5)$$

In our investigation we require the following integrals [2]:

$$\int_0^1\int_0^1 y^a(1-x)^{a-1}(1-y)^{b-1}(1-xy)^{1-a-b}\,dxdy = B(a,b), \quad (1.6)$$

Provided $\mathrm{Re}(a)>0$ and $\mathrm{Re}(b)>0$.

$$\int_0^1 x^{a-1}(1-x)^{b-1}(1-cx)^{-a-b}\,dx = (1-cx)^{-a}B(a,b) \quad (1.7)$$

Provided $\mathrm{Re}(a)>0, \mathrm{Re}(b)>0$ and $c>-1$, where $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, is the well known Beta function.

II. MAIN INTEGRAL FORMULAS

In this section ,we have proved the following integral formulas:

$$\int_0^1\int_0^1\cdots\int_0^1\int_0^1 y_1^{a_1}(1-x_1)^{a_1-1}(1-y_1)^{b_1-1}(1-x_1y_1)^{1-a_1-b_1}\times\cdots$$
$$\times\, y_r^{a_r}(1-x_r)^{a_r-1}(1-y_r)^{b_r-1}(1-x_ry_r)^{1-a_r-b_r}$$
$$\times\, P_n^{(c_1,d_1;\cdots;c_r,d_r)}\left(1-\tfrac{2y_1(1-x_1)}{1-x_1y_1},\cdots,1-\tfrac{2y_r(1-x_r)}{1-x_ry_r}\right)dx_1dy_1\cdots dx_rdy_r$$
$$=\frac{(1+c_1)_n\cdots(1+c_r)_n\,B(a_1,b_1)\cdots B(a_r,b_r)}{(n!)^r}$$
$$F_{0:2;\cdots;2}^{1:2;\cdots;2}\left[\begin{matrix}-n:1+c_1+d_1+n,a_1;\cdots;1+c_r+d_r+n,a_r;\\ -:\ 1+c_1,a_1+b_1;\cdots;\ 1+c_r,a_r+b_r;\end{matrix}1,\cdots,1\right] \quad (2.1)$$

$$\int_0^1\int_0^1\cdots\int_0^1\int_0^1 y_1^{a_1}(1-x_1)^{a_1-1}(1-y_1)^{b_1-1}(1-x_1y_1)^{1-a_1-b_1}\times\cdots$$
$$\times\, y_r^{a_r}(1-x_r)^{a_r-1}(1-y_r)^{b_r-1}(1-x_ry_r)^{1-a_r-b_r}$$
$$\times\, L_n^{(c_1,\cdots,c_r)}\left(\tfrac{y_1(1-x_1)}{1-x_1y_1},\cdots,\tfrac{y_r(1-x_r)}{1-x_ry_r}\right)dx_1dy_1\cdots dx_rdy_r$$
$$=\frac{(1+c_1)_n\cdots(1+c_r)_n\,B(a_1,b_1)\cdots B(a_r,b_r)}{(n!)^r}$$
$$F_{0:2;\cdots;2}^{1:1;\cdots;1}\left[\begin{matrix}-n:\ a_1;\cdots;\ a_r;\\ -:1+c_1,a_1+b_1;\cdots;1+c_r,a_r+b_r;\end{matrix}1,\cdots,1\right] \quad (2.2)$$

$$\int_0^1\cdots\int_0^1 x_1^{a_1-1}(1-x_1)^{b_1-1}(1+e_1x_1)^{-a_1-b_1}\times\cdots\times x_r^{a_r-1}(1-x_r)^{b_r-1}(1+e_rx_r)^{-a_r-b_r}$$

$$\times P_n^{(c_1,d_1;\cdots;c_r,d_r)}\left(1-\tfrac{2x_1}{1+e_1x_1},\cdots,1-\tfrac{2x_r}{1+e_rx_r}\right)dx_1\cdots dx_r$$

$$=\frac{(1+e_1)^{-a_1}\cdots(1+e_r)^{-a_r}(1+c_1)_n\cdots(1+c_r)_n B(a_1,b_1)\cdots B(a_r,b_r)}{(n!)^r}$$

$$F\begin{matrix}1:2;\cdots;2\\0:2;\cdots;2\end{matrix}\left[\begin{matrix}-n:1+c_1+d_1+n,a_1;\cdots;1+c_r+d_r+n,a_r;\\ -:\ 1+c_1,a_1+b_1\ ;\cdots;\ 1+c_r,a_r+b_r\ ;\end{matrix}\frac{1}{1+e_1},\cdots,\frac{1}{1+e_r}\right] \quad (2.3)$$

$$\int_0^1\cdots\int_0^1 x_1^{a_1-1}(1-x_1)^{b_1-1}(1+e_1x_1)^{-a_1-b_1}\times\cdots\times x_r^{a_r-1}(1-x_r)^{b_r-1}(1+e_rx_r)^{-a_r-b_r}$$

$$\times L_n^{(c_1,\cdots,c_r)}\left(\tfrac{x_1}{1+e_1x_1},\cdots,\tfrac{x_r}{1+e_rx_r}\right)dx_1\cdots dx_r$$

$$=\frac{(1+e_1)^{-a_1}\cdots(1+e_r)^{-a_r}(1+c_1)_n\cdots(1+c_r)_n B(a_1,b_1)\cdots B(a_r,b_r)}{(n!)^r}$$

$$F\begin{matrix}1:1;\cdots;1\\0:2;\cdots;2\end{matrix}\left[\begin{matrix}-n:\quad a_1\quad;\cdots;\quad a_r\quad;\\ -:1+c_1,a_1+b_1;\cdots;1+c_r,a_r+b_r\ ;\end{matrix}\frac{1}{1+e_1},\cdots,\frac{1}{1+e_r}\right] \quad (2.4)$$

Further, if we take $a_1=1+c_1,\cdots,a_r=1+c_r$ in (2.1), (2.2),(2.3) and (2.4) respectively we have

$$\int_0^1\int_0^1\cdots\int_0^1\int_0^1 y_1^{1+c_1}(1-x_1)^{c_1}(1-y_1)^{b_1-1}(1-x_1y_1)^{-c_1-b_1}\times\cdots$$

$$\times y_r^{1+c_r}(1-x_r)^{c_r}(1-y_r)^{b_r-1}(1-x_ry_r)^{-c_r-b_r}$$

$$\times P_n^{(c_1,d_1;\cdots;c_r,d_r)}\left(1-\tfrac{2y_1(1-x_1)}{1-x_1y_1},\cdots,1-\tfrac{2y_r(1-x_r)}{1-x_ry_r}\right)dx_1dy_1\cdots dx_rdy_r$$

$$=B(1+c_1+n,b_1)\cdots B(1+c_r+n,b_r)\,P_n^{(c_1+b_1,d_1-b_1;\cdots;c_r+b_r,d_r-b_r)}(1,\cdots,1) \quad (2.5)$$

$$\int_0^1\int_0^1\cdots\int_0^1\int_0^1 y_1^{1+c_1}(1-x_1)^{c_1}(1-y_1)^{b_1-1}(1-x_1y_1)^{-c_1-b_1}\times\cdots$$

$$\times y_r^{1+c_r}(1-x_r)^{c_r}(1-y_r)^{b_r-1}(1-x_ry_r)^{-c_r-b_r}$$

$$\times L_n^{(c_1,\cdots,c_r)}\left(\tfrac{y_1(1-x_1)}{1-x_1y_1},\cdots,\tfrac{y_r(1-x_r)}{1-x_ry_r}\right)dx_1dy_1\cdots dx_rdy_r$$

$$=B(1+c_1+n,b_1)\cdots B(1+c_r+n,b_r)\,L_n^{(c_1+b_1,\cdots,c_r+b_r)}(1,\cdots,1) \quad (2.6)$$

$$\int_0^1\cdots\int_0^1 x_1^{c_1}(1-x_1)^{b_1-1}(1+e_1x_1)^{-c_1-b_1-1}\times\cdots\times x_r^{c_r}(1-x_r)^{b_r-1}(1+e_rx_r)^{-c_r-b_r-1}$$

$$\times P_n^{(c_1,d_1;\cdots;c_r,d_r)}\left(1-\tfrac{2x_1}{1+e_1x_1},\cdots,1-\tfrac{2x_r}{1+e_rx_r}\right)dx_1\cdots dx_r$$

$$=(1+e_1)^{-c_1-1}\cdots(1+e_r)^{-c_r-1}\,B(1+c_1+n,b_1)\cdots B(1+c_r+n,b_r)$$

$$P_n^{(c_1+b_1,d_1-b_1;\cdots;c_r+b_r,d_r-b_r)}\left(\tfrac{e_1-1}{e_1+1},\cdots,\tfrac{e_r-1}{e_r+1}\right) \quad (2.7)$$

$$\int_0^1\cdots\int_0^1 x_1^{c_1}(1-x_1)^{b_1-1}(1+e_1x_1)^{-c_1-b_1-1}\times\cdots\times x_r^{c_r}(1-x_r)^{b_r-1}(1+e_rx_r)^{-c_r-b_r-1}$$

$$\times L_n^{(c_1,\cdots,c_r)}\left(\tfrac{x_1}{1+e_1x_1},\cdots,\tfrac{x_r}{1+e_rx_r}\right)dx_1\cdots dx_r$$

$$=(1+e_1)^{-c_1-1}\cdots(1+e_r)^{-c_r-1}B(1+c_1+n,b_1)\cdots B(1+c_r+n,b_r)L_n^{(c_1+b_1,\cdots,c_r+b_r)}\left(\tfrac{1}{1+e_1},\cdots,\tfrac{1}{1+e_r}\right) \quad (2.8)$$

Proof of (2.1): Denoting the left hand side of (2.1) by I, using the definition (1.1), expanding $F\begin{smallmatrix}1:1;\cdots;1\\0:1;\cdots;1\end{smallmatrix}$ in a power series and changing the order of summation and integration, we get:

$$I=\frac{(1+c_1)_n\cdots(1+c_r)_n}{(n!)^r}$$

$$\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\cdots\sum_{p_r=0}^{n-p_1-\cdots-p_r}\frac{(-n)_{p_1+\cdots+p_r}(1+c_1+d_1+n)_{p_1}\cdots(1+c_r+d_r+n)_{p_r}}{(1+c_1)_{p_1}\cdots(1+c_r)_{p_r}\,p_1!\cdots p_r!}$$

$$\times\int_0^1\int_0^1 y_1^{a_1+p_1}(1-x_1)^{a_1+p_1-1}(1-y_1)^{b_1-1}(1-x_1y_1)^{1-a_1+p_1-b_1}dx_1dy_1\times\cdots$$

$$\cdots\times\int_0^1\int_0^1 y_r^{a_r+p_r}(1-x_r)^{a_r+p_r-1}(1-y_r)^{b_r-1}(1-x_ry_r)^{1-a_r+p_r-b_r}dx_rdy_r$$

Finally, evaluating the double integral with the help of the result (1.6),we arrive after some simplification to the right hand sideof (2.1).This completes the proof of (2.1).The result (2.2) can be established similarly .The two results (2.3) and (2.4) can be established by applying the same method with the help of the result (1.7).

Remark 1.
Similar eight multiple integral formulas, involving Jacobi and Laguerre polynomials of several variables $P_n^{(c_1,d_1;\cdots;c_r,d_r)}\left(1-\frac{2(1-y_1)}{1-x_1y_1},\cdots,1-\frac{2(1-y_r)}{1-x_ry_r}\right)$, $P_n^{(c_1,d_1;\cdots;c_r,d_r)}\left(1-\frac{2(1-x_1)}{1+e_1x_1},\cdots,1-\frac{2(1-x_r)}{1+e_rx_r}\right)$ $L_n^{(c_1,\cdots,c_r)}\left(\frac{1-y_1}{1-x_1y_1},\cdots,\frac{1-y_r}{1-x_ry_r}\right)$and $L_n^{(d_1,\cdots,d_r)}\left(\frac{1-x_1}{1+c_1x_1},\cdots,\frac{1-x_r}{1+c_rx_r}\right)$,can be also obtained by applying the same method.

III. SPECIAL CASES AND APPLICATIONS

1. In (2.1) and if we take $r=2$, we get

$$\int_0^1\int_0^1\int_0^1\int_0^1 y_1^{a_1}(1-x_1)^{a_1-1}(1-y_1)^{b_1-1}(1-x_1y_1)^{1-a_1-b_1}\times$$

$$\times\, y_2^{a_2}(1-x_2)^{a_2-1}(1-y_2)^{b_2-1}(1-x_2y_2)^{1-a_2-b_2}$$

$$\times\, P_n^{(c_1,d_1;c_2,d_2)}\left(1-\frac{2y_1(1-x_1)}{1-x_1y_1},1-\frac{2y_2(1-x_2)}{1-x_2y_2}\right)dx_1dy_1dx_2dy_2$$

$$=B(a_1,b_1)\,B(a_2,b_2)\,H_n^{(c_1,d_1;c_2,d_2)}\left(a_1,a_2,a_1+b_1,a_2+b_2,1,1\right),\qquad(3.1)$$

where $H_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(\nu_1,\nu_2,p_1,p_2,x_1,x_r)$ is the generalized Rice polynomials of two variables defined by

$$H_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(\nu_1,\nu_2,p_1,p_2,x_1,x_r)=\frac{(1+\alpha_1)_n(1+\alpha_2)_n}{(n!)^2}$$

$$F\begin{matrix}1:2;2\\0:2;2\end{matrix}\left[\begin{matrix}-n:1+\alpha_1+\beta_1+n,\nu_1;1+\alpha_2+\beta_2+n,\nu_2\;;\\ -\;:\quad 1+\alpha_1,p_1\quad;\quad 1+\alpha_2,p_2\quad;\end{matrix}x_1,x_2\right]\qquad(3.2)$$

2. in (2.2), if we take $r=2, a_1=1+c_1+d_1$ and $a_2=1+c_2+d_2$, we get

$$\int_0^1\int_0^1\int_0^1\int_0^1 y_1^{1+c_1+d_1}(1-x_1)^{c_1+d_1}(1-y_1)^{b_1-1}(1-x_1y_1)^{-c_1-d_1-b_1}$$

$$\times\, y_2^{1+c_2+d_2}(1-x_2)^{c_2+d_2}(1-y_2)^{b_2-1}(1-x_2y_2)^{-c_2-d_2-b_2}$$

$$\times L_n^{(c_1,c_2)}\left(\frac{y_1(1-x_1)}{1-x_1y_1},\frac{y_2(1-x_2)}{1-x_2y_2}\right)dx_1dy_1dx_2dy_2$$

$$=\frac{(1+c_1)_n(1+c_2)_n B(1+c_1+d_1,b_1)B(1+c_2+d_2,b_2)}{(n!)^2}$$

$$Z_n^{(c_1,d_1-n;\cdots;c_2,d_2-n)}\left(c_1+d_1+b_1,1;c_2+d_2+b_2,1\right) \tag{3.3}$$

where $Z_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(b_1,x_1;b_2,x_2)$ is the generalized Batman's polynomials of two variables [1]

$$Z_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(b_1,x_1;b_2,x_2)=F\begin{matrix}1:1;1\\0:2;2\end{matrix}\left[\begin{matrix}-n:1+\alpha_1+\beta_1+n;1+\alpha_2+\beta_2+n\ ;\\ -\ :\ 1+\alpha_1,1+b_1\ \ ;\ 1+\alpha_2,1+b_2\ \ ;\end{matrix}\ x_1,x_2\right]. \tag{3.4}$$

Remark 2.

Similar two other double integrals involving Jacobi and Laguerre polynomials of two variables $P_n^{(c_1,d_1;c_2,d_2)}\left(1-\frac{2x_1}{1+e_1x_1},1-\frac{2x_2}{1+e_2x_2}\right)$ and $L_n^{(c_1,c_2)}\left(\frac{x_1}{1+e_1x_1},\frac{x_2}{1+e_2x_2}\right)$ can be also obtained as special cases of our main results (2.3) and (2.4).

3. In (2.1), if we take $r=1$, we get

$$\int_0^1\int_0^1 y^a(1-x)^{a-1}(1-y)^{b-1}(1-xy)^{1-a-b}P_n^{(c,d)}\left(1-\frac{2y(1-x)}{1-xy}\right)dxdy$$

$$=\frac{(1+c)_n B(a,b)}{n!}\,{}_3F_2\left[\begin{matrix}-n,1+c+d+n,a\ ;\\ 1+c,a+b\ ;\end{matrix}\ 1\right] \tag{3.5}$$

where $P_n^{(\alpha,\beta)}(x)$ is the classical Jacobi polynomials [6]

$$P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,1+\alpha+\beta+n\ ;\\ 1+\alpha\ ;\end{matrix}\ \frac{1-x}{2}\right] \tag{3.6}$$

In view of the definition of the generalized Rice polynomials [4]

$$H_n^{(\alpha,\beta)}(\nu,p,x)=\frac{(1+\alpha)_n}{n!}\,{}_3F_2\left[\begin{matrix}-n,1+\alpha+\beta+n,\nu\ ;\\ 1+\alpha,p\ ;\end{matrix}\ x\right], \tag{3.7}$$

the integral (3.5) can be written in the following form:

$$\int_0^1\int_0^1 y^a(1-x)^{a-1}(1-y)^{b-1}(1-xy)^{1-a-b}P_n^{(c,d)}\left(1-\frac{2y(1-x)}{1-xy}\right)dxdy$$

$$=B(a,b)\,H_n^{(c,d)}(a,a+b,1) \tag{3.8}$$

Further, in (3.5) replacing n by $2n$, putting $a=b, c=d=\frac{1}{2}h-\frac{1}{2}$ and using the following special case of the result given by [5]

$${}_3F_2\left[\begin{matrix}-2n\ ,h+2n,\ a\ ;\\ \frac{1}{2}(h+1),\ 2a\ ;\end{matrix}\ 1\right]=\frac{(\frac{1}{2})_n(\frac{1}{2}h-a+\frac{1}{2})_n}{(a+\frac{1}{2})_n(\frac{1}{2}h+\frac{1}{2})_n}, \tag{3.9}$$

we get

$$\int_0^1\int_0^1 y^a(1-x)^{a-1}(1-y)^{a-1}(1-xy)^{1-2a}P_{2n}^{(\frac{1}{2}h-\frac{1}{2},\frac{1}{2}h-\frac{1}{2})}\left(1-\frac{2y(1-x)}{1-xy}\right)dxdy$$

$$=\frac{(\frac{1}{2}h+\frac{1}{2})_{2n}B(a,a)(\frac{1}{2})_n(\frac{1}{2}h-a+\frac{1}{2})_n}{(2n)!\,(a+\frac{1}{2})_n(\frac{1}{2}h+\frac{1}{2})_n} \tag{3.10}$$

Again, in (3.5) replacing n by $2n+1$, putting $b = a+1, c = d = \frac{1}{2}h - \frac{1}{2}$ and using the result [5]

$$ {}_3F_2\left[\begin{matrix} -2n-1, h+2n+1, a & ; \\ \frac{1}{2}(h+1), 2a+1 & ; \end{matrix} 1\right] = \frac{(\frac{3}{2})_n (\frac{1}{2}h - a + \frac{1}{2})_n}{2a+1(a+\frac{3}{2})_n (\frac{1}{2}h + \frac{1}{2})_n}, \quad (3.11) $$

we get

$$ \int_0^1\int_0^1 y^a (1-x)^{a-1} (1-y)^a (1-xy)^{-2a} P_{2n+1}^{(\frac{1}{2}h-\frac{1}{2}, \frac{1}{2}h-\frac{1}{2})}\left(1 - \frac{2y(1-x)}{1-xy}\right) dxdy $$

$$ = \frac{(\frac{1}{2}h + \frac{1}{2})_{2n+1} B(a, a+1)(\frac{3}{2})_n (\frac{1}{2}h - a + \frac{1}{2})_n}{(2n+1)!\,(2a+1)(a+\frac{3}{2})_n (\frac{1}{2}h + \frac{1}{2})_n}. \quad (3.12) $$

In a similar way, a number of double integrals involving Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be also obtained, we mention here the following examples :

$$ \int_0^1\int_0^1 y^a (1-x)^{a-1} (1-y)^{a-2} (1-xy)^{2-2a} P_{2n}^{(\frac{1}{2}h-1, \frac{1}{2}h)}\left(1 - \frac{2y(1-x)}{1-xy}\right) dxdy $$

$$ = \frac{(\frac{1}{2}h)_{2n} B(a, a-1)(2a - h - 2 - 4n)(\frac{1}{2})_n (1 - a + \frac{1}{2}h)_n}{(2n)!\,(2a - h - 2)(a - \frac{1}{2})_n (\frac{1}{2}h)_n} \quad (3.13) $$

$$ \int_0^1\int_0^1 y^a (1-x)^{a-1} (1-y)^{a-2} (1-xy)^{2-2a} P_{2n+1}^{(\frac{1}{2}h-1, \frac{1}{2}h)}\left(1 - \frac{2y(1-x)}{1-xy}\right) dxdy $$

$$ = \frac{-(\frac{1}{2}h)_{2n+1} B(a, a-1)(2a + h + 4n)(\frac{3}{2})_n (\frac{1}{2}h - a + 2)_n}{a(2a-1)(2n+1)!\,(a+\frac{1}{2})_n (\frac{1}{2}h + 1)_n}. \quad (3.14) $$

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