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The Minimum Hamming Distances of the Irreducible Cyclic Codes of Length $2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}$

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Abstract: *Let $\mathbb{F}_l$ be a finite field with l elements and $n = 2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}$, where $a, a_1, a_2, \cdots, a_e$ are positive integers and $p_1, p_2, \cdots, p_e$ are distinct odd primes and $4p_1 p_2 \cdots p_e \mid l-1$. In this paper, we study the irreducible factorization of $x^{2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}} - 1$ over $\mathbb{F}_l$ and all primitive idempotents in the ring $\mathbb{F}_l[x]/\langle x^{2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}} - 1\rangle$. Moreover, we obtain the dimensions and the minimum Hamming distances of all irreducible cyclic codes of length $2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}$ over $\mathbb{F}_l$.*

Keywords: *Irreducible factorization, Primitive idempotent, Irreducible cyclic code*
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I. INTRODUCTION

Let $\mathbb{F}_l$ be a finite field with l elements, where $l = p^s$ and p is a prime. A code C over the finite field $GF(l)$ is called a cyclic if $(c_0, c_1, \cdots, c_{n-1}) \in C$ implies $(c_{n-1}, c_0, c_1, \cdots, c_{n-2}) \in C$. Cyclic codes of length n over $\mathbb{F}_l$ can be views as ideals in the ring $\mathcal{R}_n = \mathbb{F}_l[x]/\langle x^n - 1\rangle$. A cyclic code is called minimal if the corresponding ideal is minimal. In fact, many well known codes, such as BCH and Hamming codes are cyclic codes, and many other famous codes can also be constructed from cyclic codes, for example the Kerdock codes and Golay codes. Cyclic codes also have practical applications, as they can be efficiently encoded by shift registers. It is true that every cyclic code turns out to be a direct sum of some minimal cyclic codes.

Recently, a lot of papers investigate the minimal cyclic codes, see e.g. [1-6,9,18-21]. It is well known that minimal cyclic codes and primitive idempotents have a one-to-one correspondence, as every minimal cyclic code can be generated by exactly one primitive idempotent, so it is useful to determine the primitive idempotents in $\mathbb{F}_l[x]/\langle x^n - 1\rangle$. In [18], Arora and Pruthi obtained the $2n+2$ minimal cyclic codes of length in $2p^n$ over $\mathbb{F}_l$, where l is an odd prime and $\mathrm{ord}_{2p^n}(l) = \varphi(p^n)$, they also got explicit expression for the primitive idempotents, generator polynomials, minimum, distance and dimension of these codes in $\mathbb{F}_l[x]/\langle x^{2p^n} - 1\rangle$. In [19], Pruthi and Arora also studied minimal cyclic codes of length p^n over $\mathbb{F}_l$, where l is a primitive root modulo p^n, they described the l-cyclotomic cosets modulo p^n and the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{p^n} - 1\rangle$. Batra and Arora alos obtained the the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{p^n} - 1\rangle$ with $\mathrm{ord}_{p^n}(l) = \frac{\varphi(p^n)}{2}$ in [2] and [5]. In [6], Batra and Arora also obtained the the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2p^n} - 1\rangle$ where q is odd and $\mathrm{ord}_{2p^n}(l) = \frac{\varphi(2p^n)}{2}$. Sharma et al. obtained all the primitive idempotents and irreducible cyclic codes in $\mathcal{R}_n = \mathbb{F}_l[x]/\langle x^n - 1\rangle$, where $n = 2^m$, $m \geq 3$. In [4], Bakshi and Raka showed all the $3m+2$ primitive idempotents in $\mathcal{R}_n$, where $n = l_1^m l_2$, l_1, l_2 are distinct odd primes, p is a common primitive root modulo l_1^m and l_2, and $\gcd\left(\frac{\phi(l_1^m)}{2}, \frac{\phi(l_2)}{2}\right) = 1$. In [23], Singh and Pruthi presented explicit expressions for all the $4m_1m_2 + 2m_1 + 2m_2 + 1$ primitive idempotents in the ring $\mathcal{R}_n$, where $n = l_1^{m_1} l_2^{m_2}$, l_1, l_2, p are distinct odd primes, $l_1^{m_1}$ and $l_2^{m_2}$, and $\gcd\left(\phi(l_1^{m_1}), \phi(l_2^{m_2})\right) = 2$, $\mathrm{ord}_{l_1^{m_1}}(p) = \phi(l_1^{m_1})/2$ and $\mathrm{ord}_{l_2^{m_2}}(p) = \phi(l_2^{m_2})/2$. In [8], Chen et al. recursively gave minimal cyclic codes of length p^m over $\mathbb{F}_l$ where p is an odd prime and $p \mid l-1$. In [13], Fengwei Li, Qin Yue and Chengju Li gave primitive idempotents in the ring $\mathbb{F}_l[x]/\langle x^{l_1^{m_1} l_2^{m_2}} - 1\rangle$ where $m_1 \geq 1, m_2 \geq 1$ l_1, l_2 are distinct odd primes and $l_1 l_2 \mid l-1$. In [12], Fen Li and Xiwang Cao gave primitive idempotents and minimum hamming distance of all irreducible cyclic codes of length $2^a p_1^{a_1} p_2^{a_2}$ over $\mathbb{F}_l$, where a, a_1, a_2 are positive integers and p_1, p_2 are distinct odd primes and $4p_1 p_2 \mid l-1$.

In this paper, we shall generalize the results of [12]. Here we shall give all irreducible factors of $x^{2^a p_1^{\alpha_1} p_2^{\alpha_2} \dots p_e^{\alpha_e}} - 1$ over $\mathbb{F}_l$ and all primitive idempotents in the ring $\mathbb{F}_l[x]/\langle x^{2^a p_1^{\alpha_1} p_2^{\alpha_2} \dots p_e^{\alpha_e}} - 1\rangle$. In Section 2, we recall some preliminary concept about characters. In Section 3, we focus on factoring polynomials of $x^{2^a p_1^{\alpha_1} p_2^{\alpha_2} \dots p_e^{\alpha_e}} - 1$ over $\mathbb{F}_l$. In Section 4, the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^a p_1^{\alpha_1} p_2^{\alpha_2} \dots p_e^{\alpha_e}} - 1\rangle$ are given. In Section 5, the check polynomials, dimensions and minimum Hamming distances of the irreducible cyclic codes of length $2^a p_1^{\alpha_1} p_2^{\alpha_2} \dots p_e^{\alpha_e}$ generated by these primitive idempotents are obtained.

II. CHARACTERS

Let *G* be a finite abelian group of order *n*, $\mathbb{F}_l$ the finite field of order l and $n | l-1$. A group homomorphism χ from *G* into $\mathbb{F}_l^*$ is called a character of *G*. It is obvious that the set $\hat{G}$ of all the characters of *G* form a finite abelian group under the multiplication of characters.

Lemma 2.1: (The orthogonality relations for characters) (see [14, p. 189])

(1) Let χ and ψ be characters of *G*. Then

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi(g)} = \begin{cases}1 & \text{if } \chi=\psi \\ 0 & \text{if } \chi\neq\psi\end{cases};$$

(2) Furthermore, if g and h are elements of *G*, then

$$\frac{1}{|G|}\sum_{\chi\in \hat{G}}\chi(g)\overline{\chi(h)} = \begin{cases}1 & \text{if } g=h \\ 0 & \text{if } g\neq h\end{cases}.$$

The number of characters of a finite abelian group *G* is equal to $|G|$. Let *G*$=\{x_0, x_1, \cdots, x_{n-1}\}$ and $\hat{G} = \{\chi_0 = 1, \chi_1, \cdots, \chi_{n-1}\}$. Then

$$\mathbf{T} = \begin{pmatrix} \chi_0(x_0) & \chi_1(x_0) & \cdots & \chi_{n-1}(x_0) \\ \chi_0(x_1) & \chi_1(x_1) & \cdots & \chi_{n-1}(x_1) \\ \vdots & \vdots & & \vdots \\ \chi_0(x_{n-1}) & \chi_1(x_{n-1}) & \cdots & \chi_{n-1}(x_{n-1}) \end{pmatrix}$$

is called character matrix of *G* and

$$\mathbf{T}^{-1} = \frac{1}{n}\begin{pmatrix} \chi_0(x_0)^{-1} & \chi_0(x_1)^{-1} & \cdots & \chi_0(x_{n-1})^{-1} \\ \chi_1(x_0)^{-1} & \chi_1(x_1)^{-1} & \cdots & \chi_1(x_{n-1})^{-1} \\ \vdots & \vdots & & \vdots \\ \chi_{n-1}(x_0)^{-1} & \chi_{n-1}(x_1)^{-1} & \cdots & \chi_{n-1}(x_{n-1})^{-1} \end{pmatrix}.$$

III. FACTORING POLYNOMIALS OF $x^{2^a p_1^{\alpha_1} p_2^{\alpha_2} \dots p_e^{\alpha_e}} - 1$

Let $\mathbb{F}_l$ be a finite field with l elements, where $l = p^s$ and p is a prime. Every cyclic code of length *n* over a finite field $\mathbb{F}_l$ is identified with exactly one ideal of the quotient algebra $\mathbb{F}_l[x]/\langle x^n - 1\rangle$. This is one of the principal reasons why factoring $x^n - 1$ is so useful. Recently, Chen et al. [7] gave the irreducible factorization of $x^{q^t p^m} - \lambda$ over $\mathbb{F}_l$, where $l = p^s$, $\lambda \in \mathbb{F}_l^*$ and $\gcd(q, p) = 1$. The irreducible factors of $x^{2^m p^n} - 1$ and $x^{p^m} - 1$ over $\mathbb{F}_l$ were explicitly obtained in [8] and [9], respectively, where p is a prime divisor of $l-1$.

In this section, first we will give all irreducible divisors of $x^n - 1$ over $\mathbb{F}_l$ with $n = 2^a p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$, $\alpha_1\ \alpha_2, \alpha_3$ are positive integers where p_1, p_2, p_3 are distinct odd primes and $4p_1p_2p_3 | l-1$ and then generalize the results for $n = 2^a p_1^{\alpha_1} p_2^{\alpha_2} \dots p_e^{\alpha_e}$.

As usual, we adopt the notations: $k|n$ means that the integer k divides n and for a prime integer p, $p^v \parallel n$ means that $p^v | n$ but $p^{v+1} \nmid n$. There is a criterion on irreducible non-linear binomials over $\mathbb{F}_l$, which was given by Serret in 1866 (see [14, Theorem3.75] or [24, Theorem10.7]).

Lemma 3.1: Assume that $n \geq 2$. For any $a \in \mathbb{F}_l^*$ with $\operatorname{ord}(a) = k$, the binomial $x^n - a$ is irreducible over $\mathbb{F}_l$ if and only if both the following two conditions are satisfied:

(i) Every prime divisor of *n* divides k, but does not divide $\frac{l-1}{k}$;

(ii) If $4|n$, then $4|(l-1)$.

Let $\mathbb{F}_l$ be a finite field of odd order l. We denote all the non-zero elements of $\mathbb{F}_l$ by $\mathbb{F}_l^*$, i.e., the multiplicative group of $\mathbb{F}_l$. Let ξ be a primitive root of $\mathbb{F}_l^*$.

In this paper, we always assume that $2^v \parallel l-1$, $p_1^{v_1} \parallel l-1$, $p_2^{v_2} \parallel l-1$ and $p_3^{v_3} \parallel l-1$, where p_1, p_2, p_3 are distinct odd primes and v_1, v_2, v_3 are positive integers. Then $l-1 = 2^v p_1^{v_1} p_2^{v_2} p_3^{v_3} c$, where $\gcd(2p_1p_2p_3, c) = 1$.

In the following, we investigate the irreducible factorization of $x^{2^a p_1^{a_1} p_2^{a_2} p_3^{a_3}} - 1$ over $\mathbb{F}_l$. By the Division Algorithm, we have that

$$a = qv + s;\ a_1 = q_1 v_1 + s_1;\ a_2 = q_2 v_2 + s_2;\ a_3 = q_3 v_3 + s_3;$$

$0 \le s < v, 0 \le s_1 < v_1, 0 \le s_2 < v_2, 0 \le s_3 < v_3$.

For convenience, we will assume that $q \le 1, q_1 \le 1, q_2 \le 1$ and $q_3 \le 1$ in this paper. We will denote by ζ_e a primitive e-th root of unity over $\mathbb{F}_l^*$.

When $q = 0, q_1 = 0, q_2 = 0$ and $q_3 = 0$, there is an irreducible factorization over $\mathbb{F}_l$:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - 1 = \prod_{i=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3} - 1} \left(x - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}}\right).$$

Theorem 3.2: If $q = 0, q_1 = 0, q_2 = 0$ and $q_3 = 1$, then there is a factorization over $\mathbb{F}_l$:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} - 1 = \left(x^{p_3^{s_3}}\right)^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} - 1 = \prod_{i=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3} - 1} \left(x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right).$$

Moreover, suppose that $i = p_3^{t_3} k$, $\gcd(k, p_3) = 1$.

(1) If $t_3 = 0$, then the polynomial $x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ is irreducible over $\mathbb{F}_l$.

(2) If $t_3 < s_3$, then the irreducible factorization of $x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ is given as follows:

$$x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} = \prod_{j=0}^{p_3^{t_3}-1} \left(x^{p_3^{s_3-t_3}} - \zeta^j_{p_3^{t_3}} \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right). \tag{3.1}$$

(3) If $t_3 \ge s_3$, then the irreducible factorization of $x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ is given as follows:

$$x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} = \prod_{j=0}^{p_3^{s_3}-1} \left(x - \zeta^j_{p_3^{s_3}} \zeta^{k p_3^{t_3-s_3}}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right). \tag{3.2}$$

Proof: Since $\zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} \in \mathbb{F}_l$ and $s_3 < v_3$, we have

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} - 1 = \left(x^{p_3^{s_3}}\right)^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} - 1 = \prod_{i=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3} - 1} \left(x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right).$$

Suppose that $i = p_3^{t_3} k$, $\gcd(k, p_3) = 1$.

If $t_3 = 0$, then $p_3^{v_3} \parallel ord\left(\zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right)$, so $p_3 \nmid \frac{l-1}{ord\left(\zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right)}$ by $p_3^{v_3} \parallel (l-1)$. By Lemma 3.1, polynomial $x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ is irreducible over $\mathbb{F}_l$.

If $t_3 < s_3$, then there is a factorization over $\mathbb{F}_l$:

$$x^{p_3^{s_3}} - \zeta^{p_3^{t_3} k}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} = \left(x^{p_3^{s_3-t_3}}\right)^{p_3^{t_3}} - \left(\zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right)^{p_3^{t_3}} = \prod_{j=0}^{p_3^{t_3}-1} \left(x^{p_3^{s_3-t_3}} - \zeta^j_{p_3^{t_3}} \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right).$$

Since $t_3 < s_3 < v_3$ and $\gcd(k, p_3) = 1$, $p_3^{v_3} \parallel ord\left(\zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right)$ and $p_3^h \parallel ord\left(\zeta^j_{p_3^{t_3}}\right)$, where $h \le t_3 < v_3$.

Hence $p_3^{v_3} \parallel ord\left(\zeta^j_{p_3^{t_3}} \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right)$.

By $p_3^{v_3} \parallel (l-1)$ and lemma 3.1, we have, $x^{p_3^{s_3-t_3}} - \zeta^j_{p_3^{t_3}} \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ is irreducible over $\mathbb{F}_l$.

If $t_3 \ge s_3$, then the irreducible factorization of $x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ is given as follows:

$$x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} = x^{p_3^{s_3}} - \left(\zeta^{kp_3^{t_3-s_3}}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right)^{p_3^{s_3}} = \prod_{j=0}^{p_3^{s_3}-1}\left(x - \zeta^j_{p_3^{s_3}} \zeta^{kp_3^{t_3-s_3}}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right).$$

By symmetry, we get the following theorem:

Theorem 3.3: If $q = 1, q_1 = 0, q_2 = 0$ and $q_3 = 0$, then there is a factorization over $\mathbb{F}_l$:

$$x^{2^{v+s} p_1^{s_1} p_2^{s_2} p_3^{s_3}} - 1 = \left(x^{2^s}\right)^{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}} - 1 = \prod_{i=0}^{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}-1} \left(x^{2^s} - \zeta^i_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}\right).$$

Moreover, suppose that $i = 2^t k$, gcd$(k, 2)$ =1.

(1) If $t_1 = 0$, then the polynomial $x^{2^s} - \zeta^i_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}$ is irreducible over $\mathbb{F}_l$.

(2) If $t_1 < s_1$, then the irreducible factorization of $x^{2^s} - \zeta^i_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s} - \zeta^i_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}} = \prod_{j=0}^{2^t-1}\left(x^{2^{s-t}} - \zeta^j_{2^t} \zeta^k_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}\right). \qquad (3.3)$$

(3) If $t_1 \geq s_1$, then the irreducible factorization of $x^{2^s} - \zeta^i_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s} - \zeta^i_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}} = \prod_{j=0}^{2^s-1}\left(x - \zeta^j_{2^s} \zeta^{k2^{t-s}}_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}\right). \qquad (3.4)$$

Similarly we can factorize $x^{2^a p_1^{a_1} p_2^{a_2} p_3^{a_3}} - 1$ when

(i) $q = 0,\ q_1 = 1,\ q_2 = 0,\ q_3 = 0,$

(ii) $q = 0, q_1 = 0, q_2 = 1, q_3 = 0.$

Theorem 3.4: If $q = 1,\ q_1 = 1,\ q_2 = 0$ and $q_3 = 0$, then there is a factorization over $\mathbb{F}_l$:

$$x^{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} - 1 = \left(x^{2^s p_1^{s_1}}\right)^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}} - 1 = \prod_{i=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \left(x^{2^s p_1^{s_1}} - \eta^i\right),$$

where we denote $\eta = \zeta_{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}}$ as simple.

Moreover, suppose that $i = 2^t p_1^{t_1} k$, gcd$(k, 2p_1)$ =1.

When $(t, t_1) = (0,0)$, then the polynomial $x^{2^s p_1^{s_1}} - \eta^i$ is irreducible over $\mathbb{F}_l$.

When $(t, t_1) \neq (0,0)$, we consider the irreducible decomposition of $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ in the following four cases:

(1) If $t < s < v$ and $t_1 < s_1 < v_1$, then there the irreducible factorization of $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1}} - \eta^i = \prod_{j=0}^{2^t p_1^{t_1}-1}\left(x^{2^{s-t} p_1^{s_1-t_1}} - \zeta^j_{2^t p_1^{t_1}} \eta^k\right). \qquad (3.5)$$

(2) If $t \geq s$ and $t_1 < s_1 < v_1$, then the irreducible factorization of $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1}} - \eta^i = \prod_{j=0}^{2^s p_1^{t_1}-1}\left(x^{p_1^{s_1-t_1}} - \zeta^j_{2^s p_1^{t_1}} \eta^{k2^{t-s}}\right). \qquad (3.6)$$

(3) If $t < s < v$ and $t_1 \geq s_1$, then the irreducible factorization of $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1}} - \eta^i = \prod_{j=0}^{2^t p_1^{s_1}-1}\left(x^{2^{s-t}} - \zeta^j_{2^t p_1^{s_1}} \eta^{kp_1^{t_1-s_1}}\right). \qquad (3.7)$$

(4) If $t \geq s$ and $t_1 \geq s_1$, then is the irreducible factorization of $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1}} - \eta^i = \prod_{j=0}^{2^s p_1^{s_1}-1}\left(x - \zeta^j_{2^s p_1^{s_1}} \eta^{k2^{t-s} p_1^{t_1-s_1}}\right). \qquad (3.8)$$

Proof: When $(t, t_1) = (0,0)$, then $2^v \parallel ord(\eta^i)$, $p_1^{v_1} \parallel ord(\eta^i)$ and so $2 \nmid \frac{l-1}{ord(\eta^i)}$, $p_1 \nmid \frac{l-1}{ord(\eta^i)}$. By Lemma 3.1, the polynomial $x^{2^s p_1^{s_1}} - \eta^i$ is irreducible over $\mathbb{F}_l$.

When $(t, t_1) \neq (0,0)$, we consider the irreducible decomposition of $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ in the following four cases:

(1) If $t < s < v$ and $t_1 < s_1 < v_1$, then there is a factorization over $\mathbb{F}_l$:

$$x^{2^s p_1^{s_1}} - \eta^i = \left(x^{2^{s-t} p_1^{s_1 - t_1}}\right)^{2^t p_1^{t_1}} - (\eta^k)^{2^t p_1^{t_1}} = \prod_{j=0}^{2^t p_1^{t_1} - 1} \left(x^{2^{s-t} p_1^{s_1-t_1}} - \zeta^j_{2^t p_1^{t_1}} \eta^k\right).$$

Suppose that $2^h \parallel ord\left(\zeta^j_{2^t p_1^{t_1}}\right)$ and $p_1^{h_1} \parallel ord\left(\zeta^j_{2^t p_1^{t_1}}\right)$, then $h \leq t < v$ and $h_1 \leq t_1 < v_1$.

Since $2^v \parallel ord(\eta^k)$, $p_1^{v_1} \parallel ord(\eta^k)$, $2^v \parallel ord\left(\zeta^j_{2^t p_1^{t_1}} \eta^k\right)$, $p_1^{v_1} \parallel ord\left(\zeta^j_{2^t p_1^{t_1}} \eta^k\right)$, so $2 \nmid \frac{l-1}{ord\left(\zeta^j_{2^t p_1^{t_1}} \eta^k\right)}$ and $p_1 \nmid \frac{l-1}{ord\left(\zeta^j_{2^t p_1^{t_1}} \eta^k\right)}$. Hence by Lemma 3.1, the polynomial $x^{2^{s-t} p_1^{s_1-t_1}} - \zeta^j_{2^t p_1^{t_1}} \eta^k$ is irreducible over $\mathbb{F}_l$.

(2) If $t \geq s$ and $t_1 < s_1 < v_1$, then the irreducible factorization of $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1}} - \eta^i = \left(x^{p_1^{s_1-t_1}}\right)^{2^s p_1^{t_1}} - \left(\eta^{k 2^{t-s}}\right)^{2^s p_1^{t_1}} = \prod_{j=0}^{2^s p_1^{t_1} - 1} \left(x^{p_1^{s_1-t_1}} - \zeta^j_{2^s p_1^{t_1}} \eta^{k 2^{t-s}}\right).$$

(3) If $t < s < v$ and $t_1 \geq s_1$, then the irreducible factorization of $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1}} - \eta^i = \left(x^{2^{s-t}}\right)^{2^t p_1^{s_1}} - \left(\eta^{k p_1^{t_1-s_1}}\right)^{2^t p_1^{s_1}} = \prod_{j=0}^{2^t p_1^{s_1} - 1} \left(x^{2^{s-t}} - \zeta^j_{l_1^{t_1} l_2^{s_2}} \eta^{k p_1^{t_1-s_1}}\right).$$

(4) If $t \geq s$ and $t_1 \geq s_1$, then the irreducible factorization of $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1}} - \eta^i = x^{2^s p_1^{s_1}} - \left(\eta^{k 2^{t-s} p_1^{t_1-s_1}}\right)^{2^s p_1^{s_1}} = \prod_{j=0}^{2^s p_1^{s_1} - 1} \left(x - \zeta^j_{2^s p_1^{s_1}} \eta^{k 2^{t-s} p_1^{t_1-s_1}}\right).$$

This completes the proof of the theorem.

Similarly we can factorize $x^{2^a p_1^{a_1} p_2^{a_2} p_3^{a_3}} - 1$ when

(i) $q = 1,\ q_1 = 0,\ q_2 = 1,\ q_3 = 0,$

(ii) $q = 1, q_1 = 0, q_2 = 0, q_3 = 1,$

(iii) $q = 0, q_1 = 1, q_2 = 1, q_3 = 0,$

(iv) $q = 0, q_1 = 1, q_2 = 0, q_3 = 1,$

(v) $q = 0, q_1 = 0, q_2 = 1, q_3 = 1.$

Theorem 3.5: If $q = 1,\ q_1 = 1,\ q_2 = 1$ and $q_3 = 0$, then there is a factorization over $\mathbb{F}_l$:

$$x^{2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3}} - 1 = \left(x^{2^s p_1^{s_1} p_2^{s_2}}\right)^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}} - 1 = \prod_{i=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3} - 1} \left(x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i\right),$$

where we denote $\eta = \zeta_{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}}$ as simple.

Moreover, suppose that $i = 2^t p_1^{t_1} p_2^{t_2} k$, $\gcd(k, 2p_1 p_2) = 1$.

When $(t, t_1, t_2) = (0,0,0)$, then the polynomial $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ is irreducible over $\mathbb{F}_l$.

When $(t, t_1, t_2) \neq (0,0,0)$, we consider the irreducible decomposition of $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ in the following eight cases:

(1) If $t < s < v$, $t_1 < s_1 < v_1$ and $t_2 < s_2 < v_2$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i = \prod_{j=0}^{2^t p_1^{t_1} p_2^{t_2} - 1} \left(x^{2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{t_1} p_2^{t_2}} \eta^k\right). \qquad (3.9)$$

(2) If $t \geq s,\ t_1 < s_1 < v_1$ and $t_2 < s_2 < v_2$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i = \prod_{j=0}^{2^s p_1^{t_1} p_2^{t_2}-1} \left(x^{p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{t_1} p_2^{t_2}} \eta^{k 2^{t-s}} \right). \quad (3.10)$$

(3) If $t < s < v,\ t_1 \geq s_1$ and $t_2 < s_2 < v_2$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i = \prod_{j=0}^{2^t p_1^{s_1} p_2^{t_2}-1} \left(x^{2^{s-t} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{s_1} p_2^{t_2}} \eta^{k p_1^{t_1-s_1}} \right). \quad (3.11)$$

(4) If $t < s < v,\ t_1 < s_1 < v_1$ and $t_2 \geq s_2$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i = \prod_{j=0}^{2^t p_1^{t_1} p_2^{s_2}-1} \left(x^{2^{s-t} p_1^{s_1-t_1}} - \zeta^j_{2^t p_1^{t_1} p_2^{s_2}} \eta^{k p_2^{t_2-s_2}} \right). \quad (3.12)$$

(5) If $t < s < v,\ t_1 \geq s_1$ and $t_2 \geq s_2$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i = \prod_{j=0}^{2^t p_1^{s_1} p_2^{s_2}-1} \left(x^{2^{s-t}} - \zeta^j_{2^t p_1^{s_1} p_2^{s_2}} \eta^{k p_1^{t_1-s_1} p_2^{t_2-s_2}} \right). \quad (3.13)$$

(6) If $t \geq s,\ t_1 < s_1 < v_1$ and $t_2 \geq s_2$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i = \prod_{j=0}^{2^s p_1^{t_1} p_2^{s_2}-1} \left(x^{p_1^{s_1-t_1}} - \zeta^j_{2^s p_1^{t_1} p_2^{s_2}} \eta^{k 2^{t-s} p_2^{t_2-s_2}} \right). \quad (3.14)$$

(7) If $t \geq s,\ t_1 \geq s_1$ and $t_2 < s_2 < v_2$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i = \prod_{j=0}^{2^s p_1^{s_1} p_2^{t_2}-1} \left(x^{p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{s_1} p_2^{t_2}} \eta^{k 2^{t-s} p_1^{t_1-s_1}} \right). \quad (3.15)$$

(8) If $t \geq s,\ t_1 \geq s_1$ and $t_2 \geq s_2$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i = \prod_{j=0}^{2^s p_1^{s_1} p_2^{s_2}-1} \left(x - \zeta^j_{2^s p_1^{s_1} p_2^{s_2}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2}} \right). \quad (3.16)$$

Proof: Proof is similar as that of theorem 3.4.

Similarly we can factorize $x^{2^a p_1^{a_1} p_2^{a_2} p_3^{a_3}} - 1$ when

(i) $q = 1,\ q_1 = 0,\ q_2 = 1,\ q_3 = 1,$

(ii) $q = 1, q_1 = 1, q_2 = 0, q_3 = 1,$

(iii) $q = 0, q_1 = 1, q_2 = 1, q_3 = 1,$

Theorem 3.6: If $q = 1,\ q_1 = 1,\ q_2 = 1$ and $q_3 = 1$, then there is a factorization over $\mathbb{F}_l$:

$$x^{2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}} - 1 = \left(x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \right)^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}} - 1 = \prod_{i=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i \right),$$

where we denote $\eta = \zeta_{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}}$ as simple.

Moreover, suppose that $i = 2^t p_1^{t_1} p_2^{t_2} p_3^{t_3} k,\ \gcd(k, 2p_1p_2p_3) = 1.$

When $(t, t_1, t_2, t_3) = (0,0,0,0)$, then the polynomial $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ is irreducible over $\mathbb{F}_l$.

When $(t, t_1, t_2, t_3) \neq (0,0,0,0)$, we consider the irreducible decomposition of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ in the following sixteen cases:

(1) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^t p_1^{t_1} p_2^{t_2} p_3^{t_3}-1} \left(x^{2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{t_1} p_2^{t_2} p_3^{t_3}} \eta^k\right). \qquad (3.17)$$

(2) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^s p_1^{t_1} p_2^{t_2} p_3^{t_3}-1} \left(x^{p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{t_1} p_2^{t_2} p_3^{t_3}} \eta^{k 2^{t-s}}\right). \qquad (3.18)$$

(3) If $t < s < v$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^t p_1^{s_1} p_2^{t_2} p_3^{t_3}-1} \left(x^{2^{s-t} p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{s_1} p_2^{t_2} p_3^{t_3}} \eta^{k p_1^{t_1-s_1}}\right). \qquad (3.19)$$

(4) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^t p_1^{t_1} p_2^{s_2} p_3^{t_3}-1} \left(x^{2^{s-t} p_1^{s_1-t_1} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{t_1} p_2^{s_2} p_3^{t_3}} \eta^{k p_2^{t_2-s_2}}\right). \qquad (3.20)$$

(5) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^t p_1^{t_1} p_2^{t_2} p_3^{s_3}-1} \left(x^{2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{t_1} p_2^{t_2} p_3^{s_3}} \eta^{k p_3^{t_3-s_3}}\right). \qquad (3.21)$$

(6) If $t \geq s$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^s p_1^{s_1} p_2^{t_2} p_3^{t_3}-1} \left(x^{p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{s_1} p_2^{t_2} p_3^{t_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1}}\right). \qquad (3.22)$$

(7) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^s p_1^{t_1} p_2^{s_2} p_3^{t_3}-1} \left(x^{p_1^{s_1-t_1} p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{t_1} p_2^{s_2} p_3^{t_3}} \eta^{k 2^{t-s} p_2^{t_2-s_2}}\right). \qquad (3.23)$$

(8) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^s p_1^{t_1} p_2^{t_2} p_3^{s_3}-1} \left(x^{p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{t_1} p_2^{t_2} p_3^{s_3}} \eta^{k 2^{t-s} p_3^{t_3-s_3}}\right). \qquad (3.24)$$

(9) If $t < s < v$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^t p_1^{s_1} p_2^{s_2} p_3^{t_3}-1} \left(x^{2^{s-t} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{s_1} p_2^{s_2} p_3^{t_3}} \eta^{k p_1^{t_1-s_1} p_2^{t_2-s_2}}\right). \qquad (3.25)$$

(10) If $t < s < v$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^t p_1^{s_1} p_2^{t_2} p_3^{s_3}-1} \left(x^{2^{s-t} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{s_1} p_2^{t_2} p_3^{s_3}} \eta^{k p_1^{t_1-s_1} p_3^{t_3-s_3}}\right). \qquad (3.26)$$

(11) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^t p_1^{t_1} p_2^{s_2} p_3^{s_3}-1} \left(x^{2^{s-t} p_1^{s_1-t_1}} - \zeta^j_{2^t p_1^{t_1} p_2^{s_2} p_3^{s_3}} \eta^{k p_2^{t_2-s_2} p_3^{t_3-s_3}}\right). \qquad (3.27)$$

(12) If $t \geq s$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{t_3}-1} \left(x^{p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{s_1} p_2^{s_2} p_3^{t_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2}}\right). \qquad (3.28)$$

(13) If $t \geq s$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^s p_1^{s_1} p_2^{t_2} p_3^{s_3}-1} \left(x^{p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{s_1} p_2^{t_2} p_3^{s_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_3^{t_3-s_3}}\right). \qquad (3.29)$$

(14) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^s p_1^{t_1} p_2^{s_2} p_3^{s_3}-1} \left(x^{p_1^{s_1-t_1}} - \zeta^j_{2^s p_1^{t_1} p_2^{s_2} p_3^{s_3}} \eta^{k 2^{t-s} p_2^{t_2-s_2} p_3^{t_3-s_3}}\right). \qquad (3.30)$$

(15) If $t < s < v$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^t p_1^{s_1} p_2^{s_2} p_3^{s_3}-1} \left(x^{2^{s-t}} - \zeta^j_{2^t p_1^{s_1} p_2^{s_2} p_3^{s_3}} \eta^{k p_1^{t_1-s_1} p_2^{t_2-s_2} p_3^{t_3-s_3}}\right). \qquad (3.31)$$

(16) If $t \geq s$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then the irreducible factorization of $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i = \prod_{j=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}-1} \left(x - \zeta^j_{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2} p_3^{t_3-s_3}}\right). \qquad (3.32)$$

Proof: Proof is similar as that of theorem 3.4.

Now we will give all irreducible divisors of $x^n - 1$ over $\mathbb{F}_l$ with $n = 2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}$, where $a, a_1, a_2, \cdots, a_e$ are positive integers and $p_1, p_2, \cdots, p_e$ are distinct odd primes and $4p_1 p_2 \cdots p_e \mid l - 1$.

By the Division Algorithm, we have that

$$a = qv + s;\ a_1 = q_1 v_1 + s_1;\ a_2 = q_2 v_2 + s_2;\ \cdots;\ a_e = q_e v_e + s_e;$$

$0 \leq s < v, 0 \leq s_1 < v_1, 0 \leq s_2 < v_2, \cdots, 0 \leq s_e < v_e$.

When $q = q_1 = q_2 = \cdots = q_e = 0$, there is the irreducible factorization over $\mathbb{F}_l$:

$$x^{2^s p_1^{s_1} p_2^{s_2} \dots p_e^{s_e}} - 1 = \prod_{i=0}^{2^s p_1^{s_1} p_2^{s_2} \dots p_e^{s_e}-1} \left(x - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} \dots p_e^{s_e}}\right).$$

Now we have the following theorems as generalization of above results:

Theorem 3.7: If $q = 0, q_{i'} = 1$ and $q_{j'} = 0,\ 1 \leq i', j' \leq e,\ i' \neq j'$, then there is a factorization over $\mathbb{F}_l$:

$$x^{2^s \prod_{i'}\left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'}\left(p_{j'}^{s_{j'}}\right)} - 1 = \left(x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}\right)^{2^s \prod_{i'}\left(p_{i'}^{v_{i'}}\right) \prod_{j'}\left(p_{j'}^{s_{j'}}\right)} - 1$$

$$= \prod_{i=0}^{2^s \prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1} \left(x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i\right).$$

where we denote $\eta = \zeta_{2^s \prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}$ as simple.

Moreover, suppose that $i = \prod_{i'}\left(p_{i'}^{t_{i'}}\right)k$, $\gcd(k, \prod_{i'} p_{i'}) = 1$.

When $t_{i'} = 0$ for all i', then the polynomial $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ is irreducible over $\mathbb{F}_l$. When $t_{i'} \neq 0$ for all i', we consider the irreducible decomposition of $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ in the following cases:

(1) If $t_{i'} < s_{i'} < v_{i'}$ for all i', then the irreducible factorization of $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = \prod_{j=0}^{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)-1} \left(x^{\prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)} - \zeta^j_{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}\eta^k\right). \tag{3.33}$$

(2) If $t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \geq s_{v'}$, $1 \leq u', v' \leq e$, $u' \neq v' \neq j'$. then the irreducible factorization of $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = \prod_{j=0}^{\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)-1} \left(x^{\prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)} - \zeta^j_{\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)}\eta^{k\prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}\right). \tag{3.34}$$

(3) If $t_{i'} \geq s_{i'}$ for all i', then the irreducible factorization of $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = \prod_{j=0}^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)-1} \left(x - \zeta^j_{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}\eta^{k\prod_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)}\right). \tag{3.35}$$

Proof: When $t_{i'} = 0$ for all i', then $p_{i'}^{v_{i'}} \parallel ord(\eta^i)$ and $p_{i'} \nmid \frac{l-1}{ord(\eta^i)}$, $1 \leq i' \leq e$, $i' \neq j'$. By Lemma 3.1, the polynomial $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ is irreducible over $\mathbb{F}_l$.

Suppose that $i = \prod_{i'}\left(p_{i'}^{t_{i'}}\right)k$, $\gcd(k, \prod_{i'} p_{i'}) = 1$.

When $t_{i'} \neq 0$ for all i', we consider the irreducible decomposition of $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ in the following cases:

(1) If $t_{i'} < s_{i'} < v_{i'}$ for all i', then there is a factorization over $\mathbb{F}_l$:

$$x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = \left(x^{\prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)}\right)^{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)} - (\eta^k)^{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)} = \prod_{j=0}^{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)-1} \left(x^{\prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)} - \zeta^j_{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}\eta^k\right).$$

Suppose that $p_{i'}^{h_{i'}} \parallel ord\left(\zeta^j_{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}\right)$, $1 \leq i' \leq e$, $i' \neq j'$, then $h_{i'} \leq t_{i'} < v_{i'}$, $1 \leq i' \leq e$, $i' \neq j'$.

Since $p_{i'}^{v_{i'}} \parallel ord(\eta^k)$, $p_s^{v_s} \parallel ord\left(\zeta^j_{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}\eta^k\right)$, and so $p_{i'} \nmid \frac{l-1}{ord\left(\zeta^j_{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}\eta^k\right)}$, $1 \leq i' \leq e$, $i' \neq j'$. Hence by Lemma 3.1, the polynomial $x^{\prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)} - \zeta^j_{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}\eta^k$ is irreducible over $\mathbb{F}_l$.

(2) If $t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \geq s_{v'}$, $1 \leq u', v' \leq e$, $u' \neq v' \neq j'$. then the irreducible factorization of $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = \left(x^{\Pi_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)}\right)^{\Pi_{u'}\left(p_{u'}^{t_{u'}}\right)\Pi_{v'}\left(p_{v'}^{s_{v'}}\right)} - \left(\eta^{k\,\Pi_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}\right)^{\Pi_{u'}\left(p_{u'}^{t_{u'}}\right)\Pi_{v'}\left(p_{v'}^{s_{v'}}\right)}$$

$$= \prod_{j=0}^{\Pi_{u'}\left(p_{u'}^{t_{u'}}\right)\Pi_{v'}\left(p_{v'}^{s_{v'}}\right)-1} \left(x^{\Pi_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)} - \zeta^j_{\Pi_{u'}\left(p_{u'}^{t_{u'}}\right)\Pi_{v'}\left(p_{v'}^{s_{v'}}\right)} \eta^{k\,\Pi_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}\right).$$

(3) If $t_{i'} \geq s_{i'}$ for all i', then the irreducible factorization of $x^{\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = x^{\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \left(\eta^{k\,\Pi_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)}\right)^{\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} = \prod_{j=0}^{\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)-1} \left(x - \zeta^j_{\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} \eta^{k\Pi_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)}\right).$$

Theorem 3.8: If $q = 1,\ q_{i'} = 1$ and $q_{j'} = 0,\ 1 \leq i', j' \leq r,\ i' \neq j'$, then there is a factorization over $\mathbb{F}_l$:

$$x^{2^{v+s}\Pi_{i'}\left(p_{i'}^{v_{i'}+s_{i'}}\right)\Pi_{j'}\left(p_{j'}^{s_{j'}}\right)} - 1 = \left(x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)}\right)^{2^v\Pi_{i'}\left(p_{i'}^{v_{i'}}\right)\Pi_{j'}\left(p_{j'}^{s_{j'}}\right)} - 1$$

$$= \prod_{i=0}^{2^v\Pi_{i'}\left(p_{i'}^{v_{i'}}\right)\Pi_{j'}\left(p_{j'}^{s_{j'}}\right)-1} \left(x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i\right).$$

where we denote $\eta = \zeta_{2^v\Pi_{i'}\left(p_{i'}^{v_{i'}}\right)\Pi_{j'}\left(p_{j'}^{s_{j'}}\right)}$ as simple.

Moreover, suppose that $i = 2^t\prod_{i'}\left(p_{i'}^{t_{i'}}\right)k$, $\gcd(k, 2\prod_{i'} p_{i'})$ =1.

When $(t, t_{i'}) = 0$ for all i', then the polynomial $x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ is irreducible over $\mathbb{F}_l$.

When $(t, t_{i'}) \neq 0$ for all i',we consider the irreducible decomposition of $x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ in the following cases:

(1) If $t < s < v,\ t_{i'} < s_{i'} < v_{i'}$ for all i', then the irreducible factorization of $x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = \prod_{j=0}^{2^t\Pi_{i'}\left(p_{i'}^{t_{i'}}\right)-1} \left(x^{2^{s-t}\Pi_{i'}\left(i_{i'}^{s_{i'}-t_{i'}}\right)} - \zeta^j_{2^t\Pi_{i'}\left(p_{i'}^{t_{i'}}\right)}\eta^k\right). \qquad (3.36)$$

(2) If $t \geq s,\ t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \geq s_{v'}, 1 \leq u', v' \leq e,\ u' \neq v' \neq j'$. then the irreducible factorization of $x^{2^s\Pi_{i'}\left(i_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = \prod_{j=0}^{2^s\Pi_{u'}\left(p_{u'}^{t_{u'}}\right)\Pi_{v'}\left(p_{v'}^{s_{v'}}\right)-1} \left(x^{\Pi_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)} - \zeta^j_{2^s\Pi_{u'}\left(p_{u'}^{t_{u'}}\right)\Pi_{v'}\left(p_{v'}^{s_{v'}}\right)}\eta^{k2^{t-s}\Pi_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}\right).$$

(3.37)

(3) If $t < s < v,\ t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \geq s_{v'}, 1 \leq u', v' \leq e,\ u' \neq v' \neq j'$. then the irreducible factorization of $x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = \Pi_{j=0}^{2^t\Pi_{u'}\left(p_{u'}^{t_{u'}}\right)\Pi_{v'}\left(p_{v'}^{s_{v'}}\right)-1}\left(x^{2^{s-t}\Pi_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)} - \zeta^j_{2^t\Pi_{u'}\left(p_{u'}^{t_{u'}}\right)\Pi_{v'}\left(p_{v'}^{s_{v'}}\right)}\eta^{k\Pi_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}\right).$$

(3.38)

(4) If $t \geq s$ and $t_{i'} \geq s_{i'}$ for all i', then the irreducible factorization of $x^{2^s\Pi_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ is given as follows:

$$x^{2^s\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i = \prod_{j=0}^{2^s\prod_{i'}\left(p_{i'}^{s_{i'}}\right)-1}\left(x - \zeta^j_{2^s\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}\eta^{k2^{t-s}\prod_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)}\right). \qquad (3.39)$$

Proof: Proof is similar as that of theorem 3.7.

IV. PRIMITIVE IDEMPOTENTS IN $\mathbb{F}_l[x]/\langle x^{2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}} - 1\rangle$

Let $G = \langle g\rangle$ be a cyclic group of order n and g a generator of G. Suppose that $n|l-1$, then

$$\chi_i: G \to \mathbb{F}_l^*, \quad \chi_i(g^r) = \zeta_n^{ri}, 0 \le i \le n-1,$$

are group homomorphisms, where ζ_n is a primitive n-th root of unity in $\mathbb{F}_l$. They are all characters of G, i.e. $\hat{G} = \{\chi_0 = 1, \chi_1, \cdots, \chi_{n-1}\}$.

Lemma 4.1 [12]: Let $G = \langle g\rangle$ be a cyclic group of order n and $\mathbb{F}_l$ a finite field of order l. Suppose that $n|l-1$, then there are n primitive idempotents in $\mathbb{F}_l[x]/\langle x^n - 1\rangle$:

$$\theta_i(x) = \frac{1}{n}\sum_{r=0}^{n-1}\chi_i(g^{-r})x^r = \frac{1}{n}\sum_{r=0}^{n-1}\zeta_n^{-ir}x^r, \quad 0 \le i \le n-1.$$

In the following, let $e = 3$ and we always assume that $|G| = n = 2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}$, $2^v \parallel l-1$ and $p_i^{v_i} \parallel l-1, v_i \ge 1, i = 1, 2, \cdots, e$ where $p_1, p_2, \cdots, p_e$ are distinct odd primes and

$$a = qv + s;\ a_1 = q_1 v_1 + s_1;\ a_2 = q_2 v_2 + s_2;\ \cdots;\ a_e = q_e v_e + s_e;$$

$0 \le s < v, 0 \le s_1 < v_1, 0 \le s_2 < v_2, \cdots, 0 \le s_e < v_e$.

If $q = q_1 = q_2 = \cdots = q_e = 0$, i.e. $a = s < v$, $a_1 = s_1 < v_1$, $a_2 = s_2 < v_2, \cdots, a_e = s_e < v_e$, then the primitive idempotents in $\mathbb{F}_l[x]/\langle x^n - 1\rangle$ are obtained (see lemma 4.1). Hence first we shall determine the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}} - 1\rangle$ for $e = 3$ under the conditions $q \le 1$, $q_1 \le 1$, $q_2 \le 1, \cdots, q_e \le 1$ and $(q, q_1, q_2, \cdots, q_e) \ne (0,0,0,\cdots,0)$ and then determine the primitive idempotents for general case.

Theorem 4.2: In Theorem 3.2, suppose that $i = p_3^{t_3}k$, $\gcd(k, p_3) = 1$. Then all the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} - 1\rangle$ are given as follows:

(1) If $t_3 = 0$, then the irreducible polynomial $x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ corresponds to the primitive idempotent:

$$\theta_i(x) = \frac{1}{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\sum_{r=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}-1}\zeta^{-ri}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}x^{rp_3^{s_3}}.$$

(2) If $0 \ne t_3 < s_3$, then each irreducible polynomial $x^{p_3^{s_3-t_3}} - \zeta^j_{p_3^{t_3}}\zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ for $0 \le j \le p_3^{t_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+t_3}}\sum_{r_1=0}^{p_3^{t_3}-1}\sum_{r_2=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}-1}\zeta^{-jr_1}_{p_3^{t_3}}\zeta^{-r_1k-ir_2}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}x^{r_1p_3^{s_3-t_3}+r_2p_3^{s_3}}.$$

(3) If $t_3 \ge s_3$, then each irreducible polynomial $x - \zeta^j_{p_3^{s_3}}\zeta^{kp_3^{t_3-s_3}}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ for $0 \le j \le p_3^{s_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}}\sum_{r_1=0}^{p_3^{s_3}-1}\sum_{r_2=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}-1}\zeta^{-jr_1}_{p_3^{s_3}}\zeta^{-r_1kp_3^{t_3-s_3}-ir_2}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}x^{r_1+r_2p_3^{s_3}}.$$

Proof: Since $\zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ is a primitive $2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}$-th root of unity in $\mathbb{F}_l$, there is a decomposition over $\mathbb{F}_l$:

$$x^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} - 1 = \left(x^{p_3^{s_3}}\right)^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} - 1 = \prod_{i=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}-1}\left(x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right).$$

Suppose that $i = p_3^{t_3}k$, $\gcd(k, p_3) = 1$, then we divide into three cases:

(1) If $t_3 = 0$, then the polynomial $x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ is irreducible over $\mathbb{F}_l$, so $\mathbb{F}_l[x]/\langle x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\rangle$ contains the only primitive idempotent 1. For convenience, let $n = 2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}$ and $m = p_3^{s_3}$, then by the Chinese Remainder Theorem, we have the natural $\mathbb{F}_l$-algebra isomorphism:

$$\varphi: \mathbb{F}_l[x]/\langle (x^m)^n - 1\rangle \longrightarrow \mathbb{F}_l[x]/\langle x^m - \zeta_n^0\rangle \oplus \cdots \oplus \mathbb{F}_l[x]/\langle x^m - \zeta_n^{n-1}\rangle, \qquad (4.1)$$

$$\sum_{r=0}^{n-1} a_r x^{mr} \mapsto \left(\sum_{r=0}^{n-1} a_r \chi_0(g^r), \cdots, \sum_{r=0}^{n-1} a_r \chi_{n-1}(g^r)\right),$$

where $g = \overline{x^m} \in \mathbb{F}_l[x]/\langle x^{nm} - 1\rangle$ is of order n and $a_0, \cdots, a_{n-1} \in \mathbb{F}_l$.

By Lemma 4.1, the primitive idempotent corresponding to the irreducible polynomial $x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ is

$$\theta_i(x) = \frac{1}{n}\sum_{r=0}^{n-1} \chi_i(g^{-r}) x^{rm} = \frac{1}{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} \sum_{r=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}-1} \zeta^{-ri}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} x^{r p_3^{s_3}}.$$

(2) If $0 \neq t_3 < s_3$, then by Theorem 3.2, there is an irreducible factorization over $\mathbb{F}_l$:

$$x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} = \left(x^{p_3^{s_3-t_3}}\right)^{p_3^{t_3}} - \left(\zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right)^{p_3^{t_3}} = \prod_{j=0}^{p_3^{t_3}-1}\left(x^{p_3^{s_3-t_3}} - \zeta^j_{p_3^{t_3}} \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right).$$

First, we find all the primitive idempotents in the ring $\mathbb{F}_l[x]/\langle x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\rangle$. It is clear that the prinicipal ideal $\langle x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\rangle = \langle \left(x^{p_3^{s_3-t_3}} / \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right)^{p_3^{t_3}} - 1\rangle$. In (4.1), let $n = p_3^{t_3}$ and x^m be replaced by $x^{p_3^{s_3-t_3}} / \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$. Similarly, each primitive idempotent of $\mathbb{F}_l[x]/\langle x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\rangle$ corresponding to each irreducible polynomial $x^{p_3^{s_3-t_3}} - \zeta^j_{p_3^{t_3}} \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ for $0 \leq j \leq p_3^{t_3} - 1$ is

$$\vartheta_{i,j}(x) = \frac{1}{p_3^{t_3}} \sum_{r=0}^{p_3^{t_3}-1} \chi_j(g^{-r}) \left(x^{p_3^{s_3-t_3}} / \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\right)^r = \frac{1}{p_3^{t_3}} \sum_{r=0}^{p_3^{t_3}-1} \zeta^{-rj}_{p_3^{t_3}} \zeta^{-kr}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} x^{r p_3^{s_3-t_3}}.$$

Second, we lift each primitive idempotent $\vartheta_{i,j}$ for $0 \leq j \leq p_3^{t_3} - 1$ in $\mathbb{F}_l[x]/\langle x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\rangle$ to a primitive idempotent in $\mathbb{F}_l[x]/\langle x^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} - 1\rangle$. In (4.1), $e_{i,j} = (0, \cdots, \vartheta_{i,j}(x), \cdots, 0)$ is a primitive idempotent in $\mathbb{F}_l[x]/\langle x^m - \zeta_n^0\rangle \oplus \cdots \oplus \mathbb{F}_l[x]/\langle x^m - \zeta_n^{n-1}\rangle$. Suppose that $\varphi\left(\theta_{i,j}(x)\right) = e_{i,j}$, $\theta_{i,j}(x) = \sum_{r=0}^{n-1} a_r x^{mr} \in \mathbb{F}_l[x]/\langle x^{mn} - 1\rangle$. By Lemma 4.1, we have

$$(a_0, a_1, \cdots, a_{n-1}) = e_{i,j}\,\mathbf{T}^{-1} = \frac{1}{n}\left(\vartheta_{i,j}\chi_i(g^0)^{-1}, \cdots, \vartheta_{i,j}\chi_i(g^{n-1})^{-1}\right).$$

Hence each primitive idempotent in $\mathbb{F}_l[x]/\langle x^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} - 1\rangle$ corresponding to each irreducible polynomial $x^{p_3^{s_3-t_3}} - \zeta^j_{p_3^{t_3}} \zeta^k_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ is

$$\theta_{i,j}(x) = \frac{1}{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+t_3}} \sum_{r_1=0}^{p_3^{t_3}-1} \sum_{r_2=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}-1} \zeta^{-jr_1}_{p_3^{t_3}} \zeta^{-r_1 k - i r_2}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} x^{r_1 p_3^{s_3-t_3} + r_2 p_3^{s_3}}, \quad 0 \leq j \leq p_3^{t_3} - 1.$$

(3) If $t_3 \geq s_3$, then similarly, we know all primitive idempotents in $\mathbb{F}_l[x]/\langle x^{p_3^{s_3}} - \zeta^i_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}\rangle$:

$$\vartheta_{i,j}(x) = \frac{1}{p_3^{s_3}} \sum_{r=0}^{p_3^{s_3}-1} \zeta^{-rj}_{p_3^{s_3}} \zeta^{-rkp_3^{t_3-s_3}}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} x^r, \quad 0 \leq j \leq p_3^{s_3} - 1.$$

Hence each primitive idempotent in $\mathbb{F}_l[x]/\langle x^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} - 1\rangle$ corresponding to each irreducible polynomial $x - \zeta^j_{p_3^{s_3}} \zeta^{kp_3^{t_3-s_3}}_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}$ over $\mathbb{F}_l$ is

$$\theta_{i,j}(x) = \frac{1}{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} \sum_{r_1=0}^{p_3^{s_3}-1} \sum_{r_2=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}-1} \zeta_{l_3^{s_3}}^{-jr_1} \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{-r_1 k p_3^{t_3-s_3} - ir_2} x^{r_1 + r_2 p_3^{s_3}}.$$

Remark 4.3: Suppose that $s = s_1 = s_2 = 0$, then it is just [9, Theorem 3.4].

By symmetry, we get the following theorem:

Theorem 4.4: In Theorem 3.3, suppose that $i = 2^t k$, gcd$(k, 2)$ =1. Then all the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^{v+s} p_1^{s_1} p_2^{s_2} p_3^{s_3}} - 1\rangle$ are given as follows:

(1) If $t = 0$, then the irreducible polynomial $x^{2^s} - \zeta_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}^{i}$ over $\mathbb{F}_l$ corresponds to the primitive idempotent:

$$\theta_i(x) = \frac{1}{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}} \sum_{r=0}^{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}^{-ri} x^{r2^s}.$$

(2) If $0 \neq t < s$, then each irreducible polynomial $x^{2^{s-t}} - \zeta_{2^t}^{j} \zeta_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}^{k}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{s_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^t-1} \sum_{r_2=0}^{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^t}^{-jr_1} \zeta_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}^{-r_1 k - ir_2} x^{r_1 2^{s-t} + r_2 2^s}.$$

(3) If $t_1 \geq s_1$, then each irreducible polynomial $x - \zeta_{2^s}^{j} \zeta_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}^{k2^{t-s}}$ over $\mathbb{F}_l$ for $0 \leq j \leq l_1^{s_1} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{s_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^s-1} \sum_{r_2=0}^{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^s}^{-jr_1} \zeta_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}^{-r_1 k 2^{t-s} - ir_2} x^{r_1 + r_2 2^s}.$$

Similarly, we can obtain primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^s p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} - 1\rangle$ and $\mathbb{F}_l[x]/\langle x^{2^s p_1^{s_1} p_2^{v_2+s_2} p_3^{s_3}} - 1\rangle$.

If $q = 1$, $q_1 = 1$, $q_2 = 0$ and $q_3 = 0$, then there is a factorization in $\mathbb{F}_l[x]$:

$$x^{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} - 1 = \left(x^{2^s p_1^{s_1}}\right)^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}} - 1 = \prod_{i=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \left(x^{2^s p_1^{s_1}} - \eta^i\right), \qquad (4.2)$$

where $s < v$, $s_1 < v_1$ and $\eta = \zeta_{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}}$ is a primitive $2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}$-th root of unity over $\mathbb{F}_l$.

Theorem 4.5: In Theorem 3.4, suppose that $i = 2^t p_1^{t_1} k$, gcd$(k, 2p_1)$ =1. Then all the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} - 1\rangle$ are given as follows:

When $(t, t_1) = (0,0)$, each irreducible polynomial $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ corresponds to the primitive idempotent:

$$\theta_i(x) = \frac{1}{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}} \sum_{r=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \eta^{-ri} x^{r2^s p_1^{s_1}}.$$

When $(t, t_1) \neq (0,0)$, we divide into four cases:

(1) If $t < s < v$ and $t_1 < s_1 < v_1$, then each irreducible polynomial $x^{2^{s-t} p_1^{s_1-t_1}} - \zeta_{2^t p_1^{t_1}}^{j} \eta^k$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{t_1} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+t_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^t p_1^{t_1}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^t p_1^{t_1}}^{-jr_1} \eta^{-r_1 k - ir_2} x^{r_1 2^{s-t} p_1^{s_1-t_1} + r_2 2^s p_1^{s_1}}.$$

(2) If $t \geq s$ and $t_1 < s_1 < v_1$, then each irreducible polynomial $x^{p_1^{s_1-t_1}} - \zeta_{2^s p_1^{t_1}}^{j} \eta^{k2^{t-s}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{t_1} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+t_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^s p_1^{t_1}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^s p_1^{t_1}}^{-jr_1} \eta^{-r_1 k 2^{t-s} - ir_2} x^{r_1 p_1^{s_1-t_1} + r_2 2^s p_1^{s_1}}.$$

(3) If $t < s < v$ and $t_1 \geq s_1$, then each irreducible polynomial $x^{2^{s-t}} - \zeta_{2^t p_1^{s_1}}^j \eta^{k p_1^{t_1-s_1}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{s_1} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^t p_1^{s_1}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^t p_1^{s_1}}^{-jr_1} \eta^{-r_1 k p_1^{t_1-s_1} - ir_2} x^{r_1 2^{s-t} + r_2 2^s p_1^{s_1}}.$$

(4) If $t \geq s$ and $t_1 \geq s_1$, then each irreducible polynomial $x - \zeta_{2^s p_1^{s_1}}^j \eta^{k 2^{t-s} p_1^{t_1-s_1}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{s_1} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^s p_1^{s_1}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^s p_1^{s_1}}^{-jr_1} \eta^{-r_1 k 2^{t-s} p_1^{t_1-s_1} - ir_2} x^{r_1 + r_2 2^s p_1^{s_1}}.$$

Proof: By (4.2), suppose that $i = 2^t p_1^{t_1} k$, $\gcd(k, 2p_1) = 1$.

When $(t, t_1) = (0,0)$, then the polynomial $x^{2^s p_1^{s_1}} - \eta^i$ is irreducible over $\mathbb{F}_l$, so $\mathbb{F}_l[x]/ < x^{2^s p_1^{s_1}} - \eta^i >$ contains the only primitive idempotent 1. For convenience, let $n = 2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}$ and $m = 2^s p_1^{s_1}$, then by the Chinese Remainder Theorem, we have the natural $\mathbb{F}_l$-algebra isomorphism:

$$\varphi: \mathbb{F}_l[x]/\langle (x^m)^n - 1 \rangle \longrightarrow \mathbb{F}_l[x]/\langle x^m - \eta^0 \rangle \oplus \cdots \oplus \mathbb{F}_l[x]/\langle x^m - \eta^{n-1} \rangle, \quad (4.3)$$

$$\sum_{r=0}^{n-1} a_r x^{mr} \mapsto \left(\sum_{r=0}^{n-1} a_r \chi_0(g^r), \cdots, \sum_{r=0}^{n-1} a_r \chi_{n-1}(g^r) \right),$$

where $g = \overline{x^m} \in \mathbb{F}_l[x]/\langle x^{nm} - 1 \rangle$ is of order n and $a_0, \cdots, a_{n-1} \in \mathbb{F}_l$.

By Lemma 4.1, the primitive idempotent corresponding to the irreducible polynomial $x^{2^s p_1^{s_1}} - \eta^i$ over $\mathbb{F}_l$ is:

$$\theta_i(x) = \frac{1}{n} \sum_{r=0}^{n-1} \chi_i(g^{-r}) x^{rm} = \frac{1}{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}} \sum_{r=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \eta^{-ri} x^{r 2^s p_1^{s_1}}.$$

When $(t, t_1) \neq (0,0)$, we divide into four cases:

(1) If $t < s < v$ and $t_1 < s_1 < v_1$, then by Theorem 3.4, there is an irreducible factorization over $\mathbb{F}_l$:

$$x^{2^s p_1^{s_1}} - \eta^i = \prod_{j=0}^{2^t p_1^{t_1}-1} \left(x^{2^{s-t} p_1^{s_1-t_1}} - \zeta_{2^t p_1^{t_1}}^j \eta^k \right).$$

First we find all the primitive idempotents in the ring $\mathbb{F}_l[x]/\langle x^{2^s p_1^{s_1}} - \eta^i \rangle$. It is clear that the prinicipal ideal $\langle x^{2^s p_1^{s_1}} - \eta^i \rangle = \langle \left(x^{2^{s-t} p_1^{s_1-t_1}} / \eta^k \right)^{2^t p_1^{t_1}} - 1 \rangle$. In (4.3), let $n = 2^t p_1^{t_1}$ and x^m be replaced by $x^{2^{s-t} p_1^{s_1-t_1}} / \eta^k$. Similarly, each primitive idempotent of $\mathbb{F}_l[x]/\langle x^{2^s p_1^{s_1}} - \eta^i \rangle$ corresponding to each irreducible polynomial $x^{2^{s-t} p_1^{s_1-t_1}} - \zeta_{2^t p_1^{t_1}}^j \eta^k$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{t_1} - 1$ is

$$\vartheta_{i,j}(x) = \frac{1}{2^t p_1^{t_1}} \sum_{r=0}^{2^t p_1^{t_1}-1} \chi_j(g^{-r}) \left(x^{2^{s-t} p_1^{s_1-t_1}} / \eta^k \right)^r = \frac{1}{2^t p_1^{t_1}} \sum_{r=0}^{2^t p_1^{t_1}-1} \zeta_{2^t p_1^{t_1}}^{-rj} \eta^{-kr} x^{r 2^{s-t} p_1^{s_1-t_1}}.$$

Second, we lift each primitive idempotent $\vartheta_{i,j}$ for $0 \leq j \leq 2^t p_1^{t_1} - 1$ in $\mathbb{F}_l[x]/ < x^{2^s p_1^{s_1}} - \eta^i >$ to a primitive idempotent in $\mathbb{F}_l[x]/ < \langle x^{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} - 1 \rangle$. In (4.3), $e_{i,j} = (0, \cdots, \vartheta_{i,j}(x), \cdots, 0)$ is a primitive idempotent in $\mathbb{F}_l[x]/\langle x^m - \eta^0 \rangle \oplus \cdots \mathbb{F}_l[x]/\langle x^m - \eta^{n-1} \rangle$. Suppose that $\varphi\left(\theta_{i,j}(x)\right) = e_{i,j}$, $\theta_{i,j}(x) = \sum_{r=0}^{n-1} a_r x^{mr} \in \mathbb{F}_l[x]/\langle x^{mn} - 1 \rangle$. By Lemma 4.1, we have

$$(a_0, a_1, \cdots, a_{n-1}) = e_{i,j} \mathbf{T}^{-1} = \frac{1}{n} \left(\vartheta_{i,j} \chi_i(g^0)^{-1}, \cdots, \vartheta_{i,j} \chi_i(g^{n-1})^{-1} \right).$$

Hence each primitive idempotent in $\mathbb{F}_l[x]/\langle x^{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} - 1 \rangle$ corresponding to each irreducible polynomial $x^{2^{s-t} p_1^{s_1-t_1}} - \zeta_{2^t p_1^{t_1}}^j \eta^k$, $0 \leq j \leq 2^t p_1^{t_1} - 1$, over $\mathbb{F}_l$ is

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+t_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^t p_1^{t_1}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^t p_1^{t_1}}^{-jr_1} \eta^{-r_1 k - ir_2} x^{r_1 2^{s-t} p_1^{s_1-t_1} + r_2 2^s p_1^{s_1}}.$$

(2) If $t \geq s$ and $t_1 < s_1 < v_1$, then by Theorem 3.4, there is an irreducible factorization over $\mathbb{F}_l$:

$$x^{2^s p_1^{s_1}} - \eta^i = \prod_{j=0}^{2^s p_1^{t_1}-1} \left(x^{p_1^{s_1-t_1}} - \zeta_{2^s p_1^{t_1}}^{j} \eta^{k 2^{t-s}} \right).$$

Similarly, we know all primitive idempotents in $\mathbb{F}_l\,[x]/\langle x^{2^s p_1^{s_1}} - \eta^i \rangle$:

$$\vartheta_{i,j}(x) = \frac{1}{2^s p_1^{t_1}} \sum_{r=0}^{2^s p_1^{t_1}-1} \zeta_{2^s p_1^{t_1}}^{-rj} \eta^{-kr 2^{t-s}} x^{r p_1^{s_1-t_1}}, \quad 0 \leq j \leq 2^s p_1^{t_1} - 1.$$

Hence each primitive idempotent in $\mathbb{F}_l\,[x]/\langle x^{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} - 1 \rangle$ corresponding to each irreducible polynomial $x^{p_1^{s_1-t_1}} - \zeta_{2^s p_1^{t_1}}^{j} \eta^{k 2^{t-s}}$, $0 \leq j \leq 2^s p_1^{t_1} - 1$, over $\mathbb{F}_l$ is

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+t_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^s p_1^{t_1}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^s p_1^{t_1}}^{-jr_1} \eta^{-r_1 k 2^{t-s} - ir_2} x^{r_1 p_1^{s_1-t_1} + r_2 2^s p_1^{s_1}}.$$

(3) If $t < s < v$ and $t_1 \geq s_1$, then by Theorem 3.4, there is an irreducible factorization over $\mathbb{F}_l$:

$$x^{2^s p_1^{s_1}} - \eta^i = \prod_{j=0}^{2^t p_1^{s_1}-1} \left(x^{2^{s-t}} - \zeta_{2^t p_1^{s_1}}^{j} \eta^{k p_1^{t_1-s_1}} \right).$$

Similarly, we know all primitive idempotents in $\mathbb{F}_l\,[x]/\langle x^{2^s p_1^{s_1}} - \eta^i \rangle$:

$$\vartheta_{i,j}(x) = \frac{1}{2^t p_1^{s_1}} \sum_{r=0}^{2^t p_1^{s_1}-1} \zeta_{2^t p_1^{s_1}}^{-rj} \eta^{-kr p_1^{t_1-s_1}} x^{r 2^{s-t}}, \quad 0 \leq j \leq 2^t p_1^{s_1} - 1.$$

Hence each primitive idempotent in $\mathbb{F}_l\,[x]/\langle x^{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} - 1 \rangle$ corresponding to each irreducible polynomial $x^{2^{s-t}} - \zeta_{2^t p_1^{s_1}}^{j} \eta^{k p_1^{t_1-s_1}}$, $0 \leq j \leq 2^t p_1^{s_1} - 1$, over $\mathbb{F}_l$ is

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^t p_1^{s_1}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^t p_1^{s_1}}^{-jr_1} \eta^{-r_1 k p_1^{t_1-s_1} - ir_2} x^{r_1 2^{s-t} + r_2 2^s p_1^{s_1}}.$$

(4) If $t_1 \geq s_1$ and $t_2 \geq s_2$, then by Theorem 3.4, there is an irreducible factorization over $\mathbb{F}_l$:

$$x^{2^s p_1^{s_1}} - \eta^i = \prod_{j=0}^{2^s p_1^{s_1}-1} \left(x - \zeta_{2^s p_1^{s_1}}^{j} \eta^{k 2^{t-s} p_1^{t_1-s_1}} \right).$$

Similarly, we know all primitive idempotents in $\mathbb{F}_l\,[x]/\langle x^{2^s p_1^{s_1}} - \eta^i \rangle$:

$$\vartheta_{i,j}(x) = \frac{1}{2^s p_1^{s_1}} \sum_{r=0}^{2^s p_1^{s_1}-1} \zeta_{2^s p_1^{s_1}}^{-rj} \eta^{-kr 2^{t-s} p_1^{t_1-s_1}} x^{r}, \quad 0 \leq j \leq 2^s p_1^{s_1} - 1.$$

Hence each primitive idempotent in $\mathbb{F}_l\,[x]/\langle x^{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} - 1 \rangle$ corresponding to each irreducible polynomial $x - \zeta_{2^s p_1^{s_1}}^{j} \eta^{k 2^{t-s} p_1^{t_1-s_1}}$, $0 \leq j \leq 2^s p_1^{s_1} - 1$, over $\mathbb{F}_l$ is

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}} \sum_{r_1=0}^{2^s p_1^{s_1}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}-1} \zeta_{2^s p_1^{s_1}}^{-jr_1} \eta^{-r_1 k 2^{t-s} p_1^{t_1-s_1} - ir_2} x^{r_1 + r_2 l_1^{s_1} l_2^{s_2}}.$$

Similarly we can find primitive idempotents in in $\mathbb{F}_l\,[x]/\langle x^{2^a p_1^{a_1} p_2^{a_2} p_3^{a_3}} - 1 \rangle$ when

(i) $q = 1,\ q_1 = 0,\ q_2 = 1,\ q_3 = 0,$

(ii) $q = 1, q_1 = 0, q_2 = 0, q_3 = 1,$

(iii) $q = 0, q_1 = 1, q_2 = 1, q_3 = 0,$

(iv) $q = 0, q_1 = 1, q_2 = 0, q_3 = 1,$
(v) $q = 0, q_1 = 0, q_2 = 1, q_3 = 1.$

By the same method, we get the following theorems:

Theorem 4.6: In Theorem 3.5, suppose that $i = 2^t p_1^{t_1} p_2^{t_2} k$, $\gcd(k, 2p_1 p_2)$ =1 and $\eta = \zeta_{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}}$ is a primitive $2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}$-th root of unity over $\mathbb{F}_l$.

Then all the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3}} - 1\rangle$ are given as follows:

When $(t, t_1, t_2) = (0,0,0)$, each irreducible polynomial $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$ over $\mathbb{F}_l$ corresponds to the primitive idempotent:

$$\theta_i(x) = \frac{1}{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}} \sum_{r=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}-1} \eta^{-ri} x^{r 2^s p_1^{s_1} p_2^{s_2}}.$$

When $(t, t_1, t_2) \neq (0,0,0)$, we divide into eight cases:

(1) If $t < s < v$, $t_1 < s_1 < v_1$ and $t_2 < s_2 < v_2$, then each irreducible polynomial $x^{2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta_{2^t p_1^{t_1} p_2^{t_2}}^j \eta^k$ over $\mathbb{F}_l$ for $0 \le j \le 2^t p_1^{t_1} p_2^{t_2} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{s_3}} \sum_{r_1=0}^{2^t p_1^{t_1} p_2^{t_2}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}-1} \zeta_{2^t p_1^{t_1} p_2^{t_2}}^{-jr_1} \eta^{-r_1 k - i r_2} x^{r_1 2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2} + r_2 2^s p_1^{s_1} p_2^{s_2}}.$$

(2) If $t \ge s$, $t_1 < s_1 < v_1$ and $t_2 < s_2 < v_2$, then each irreducible polynomial $x^{p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta_{2^s p_1^{t_1} p_2^{t_2}}^j \eta^{k 2^{t-s}}$ over $\mathbb{F}_l$ for $0 \le j \le 2^s p_1^{t_1} p_2^{t_2} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{s_3}} \sum_{r_1=0}^{2^s p_1^{t_1} p_2^{t_2}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}-1} \zeta_{2^s p_1^{t_1} p_2^{t_2}}^{-jr_1} \eta^{-r_1 k 2^{t-s} - i r_2} x^{r_1 p_1^{s_1-t_1} p_2^{s_2-t_2} + r_2 2^s p_1^{s_1} p_2^{s_2}}.$$

(3) If $t < s < v$, $t_1 \ge s_1$ and $t_2 < s_2 < v_2$, then each irreducible polynomial $x^{2^{s-t} p_2^{s_2-t_2}} - \zeta_{2^t p_1^{s_1} p_2^{t_2}}^j \eta^{k p_1^{t_1-s_1}}$ over $\mathbb{F}_l$ for $0 \le j \le 2^t p_1^{s_1} p_2^{t_2} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{s_3}} \sum_{r_1=0}^{2^t p_1^{s_1} p_2^{t_2}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}-1} \zeta_{2^t p_1^{s_1} p_2^{t_2}}^{-jr_1} \eta^{-r_1 k p_1^{t_1-s_1} - i r_2} x^{r_1 2^{s-t} p_2^{s_2-t_2} + r_2 2^s p_1^{s_1} p_2^{s_2}}.$$

(4) If $t < s < v$, $t_1 < s_1 < v_1$ and $t_2 \ge s_2$, then each irreducible polynomial $x^{2^{s-t} p_1^{s_1-t_1}} - \zeta_{2^t p_1^{t_1} p_2^{s_2}}^j \eta^{k p_2^{t_2-s_2}}$ over $\mathbb{F}_l$ for $0 \le j \le 2^t p_1^{t_1} p_2^{s_2} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{s_3}} \sum_{r_1=0}^{2^t p_1^{t_1} p_2^{s_2}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}-1} \zeta_{2^t p_1^{t_1} p_2^{s_2}}^{-jr_1} \eta^{-r_1 k p_2^{t_2-s_2} - i r_2} x^{r_1 2^{s-t} p_1^{s_1-t_1} + r_2 2^s p_1^{s_1} p_2^{s_2}}.$$

(5) If $t < s < v$, $t_1 \ge s_1$ and $t_2 \ge s_2$, then each irreducible polynomial $x^{2^{s-t}} - \zeta_{2^t p_1^{s_1} p_2^{s_2}}^j \eta^{k p_1^{t_1-s_1} p_2^{t_2-s_2}}$ over $\mathbb{F}_l$ for $0 \le j \le 2^t p_1^{s_1} p_2^{s_2} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3}} \sum_{r_1=0}^{2^t p_1^{s_1} p_2^{s_2}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}-1} \zeta_{2^t p_1^{s_1} p_2^{s_2}}^{-jr_1} \eta^{-r_1 k p_1^{t_1-s_1} p_2^{t_2-s_2} - i r_2} x^{r_1 2^{s-t} + r_2 2^s p_1^{s_1} p_2^{s_2}}.$$

(6) If $t \ge s$, $t_1 < s_1 < v_1$ and $t_2 \ge s_2$, then each irreducible polynomial $x^{p_1^{s_1-t_1}} - \zeta_{2^s p_1^{t_1} p_2^{s_2}}^j \eta^{k 2^{t-s} p_2^{t_2-s_2}}$ over $\mathbb{F}_l$ for $0 \le j \le 2^s p_1^{t_1} p_2^{s_2} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{s_3}} \sum_{r_1=0}^{2^s p_1^{t_1} p_2^{s_2}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}-1} \zeta_{2^s p_1^{t_1} p_2^{s_2}}^{-jr_1} \eta^{-r_1 k 2^{t-s} p_2^{t_2-s_2} - i r_2} x^{r_1 p_1^{s_1-t_1} + r_2 2^s p_1^{s_1} p_2^{s_2}}.$$

(7) If $t \geq s$, $t_1 \geq s_1$ and $t_2 < s_2 < v_2$, then each irreducible polynomial $x^{p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{s_1} p_2^{t_2}} \eta^{k2^{t-s} p_1^{t_1-s_1}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{s_1} p_2^{t_2} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{s_3}} \sum_{r_1=0}^{2^s p_1^{s_1} p_2^{t_2}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}-1} \zeta^{-jr_1}_{2^s p_1^{s_1} p_2^{t_2}} \eta^{-r_1 k 2^{t-s} p_1^{t_1-s_1} - ir_2} x^{r_1 p_2^{s_2-t_2} + r_2 2^s p_1^{s_1} p_2^{s_2}}.$$

(8) If $t \geq s$, $t_1 \geq s_1$ and $t_2 \geq s_2$, then each irreducible polynomial $x - \zeta^j_{2^s p_1^{s_1} p_2^{s_2}} \eta^{k2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2}}$ over $\mathbb{F}_l$ for $0 \leq j \leq l_1^{s_1} l_2^{s_2} l_3^{s_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3}} \sum_{r_1=0}^{2^s p_1^{s_1} p_2^{s_2}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}-1} \zeta^{-jr_1}_{2^s p_1^{s_1} p_2^{s_2}} \eta^{-r_1 k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2} - ir_2} x^{r_1 + r_2 2^s p_1^{s_1} p_2^{s_2}}.$$

Theorem 4.7: In Theorem 3.6, suppose that $i = 2^t p_1^{t_1} p_2^{t_2} p_3^{t_3} k$, $\gcd(k, 2p_1 p_2 p_3) = 1$ and $\eta = \zeta_{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}}$ is a primitive $2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}$-th root of unity over $\mathbb{F}_l$.

Then all the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}} - 1\rangle$ are given as follows:

When $(t, t_1, t_2, t_3) = (0,0,0,0)$, each irreducible polynomial $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$ over $\mathbb{F}_l$ corresponds to the primitive idempotent:

$$\theta_i(x) = \frac{1}{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}} \sum_{r=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \eta^{-ri} x^{r 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}}.$$

When $(t, t_1, t_2, t_3) \neq (0,0,0,0)$, we divide into sixteen cases:

(1) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then each irreducible polynomial $x^{2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{t_1} p_2^{t_2} p_3^{t_3}} \eta^k$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{t_1} p_2^{t_2} p_3^{t_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{v_3+t_3}} \left\{ \sum_{r_1=0}^{2^t p_1^{t_1} p_2^{t_2} p_3^{t_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(\zeta^{-jr_1}_{2^t p_1^{t_1} p_2^{t_2} p_3^{t_3}} \eta^{-r_1 k - ir_2} x^{r_1 2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \right) \right\}$$

(2) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then each irreducible polynomial $x^{p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{t_1} p_2^{t_2} p_3^{t_3}} \eta^{k2^{t-s}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{t_1} p_2^{t_2} p_3^{t_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{v_3+t_3}} \left\{ \sum_{r_1=0}^{2^s p_1^{t_1} p_2^{t_2} p_3^{t_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(\zeta^{-jr_1}_{2^s p_1^{t_1} p_2^{t_2} p_3^{t_3}} \eta^{-r_1 k 2^{t-s} - ir_2} x^{r_1 p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \right) \right\}$$

(3) If $t < s < v$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then each irreducible polynomial $x^{2^{s-t} p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{s_1} p_2^{t_2} p_3^{t_3}} \eta^{kp_1^{t_1-s_1}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{s_1} p_2^{t_2} p_3^{t_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{v_3+t_3}} \left\{ \sum_{r_1=0}^{2^t p_1^{s_1} p_2^{t_2} p_3^{t_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(\zeta^{-jr_1}_{2^t p_1^{s_1} p_2^{t_2} p_3^{t_3}} \eta^{-r_1 k p_1^{t_1-s_1} - ir_2} x^{r_1 2^{s-t} p_2^{s_2-t_2} p_3^{s_3-t_3} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \right) \right\}$$

(4) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then each irreducible polynomial $x^{2^{s-t} p_1^{s_1-t_1} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{t_1} p_2^{s_2} p_3^{t_3}} \eta^{k p_2^{t_2-s_2}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{t_1} p_2^{s_2} p_3^{t_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{v_3+t_3}} \left\{ \sum_{r_1=0}^{2^t p_1^{t_1} p_2^{s_2} p_3^{t_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(\zeta^{-j r_1}_{2^t p_1^{t_1} p_2^{s_2} p_3^{t_3}} \eta^{-r_1 k p_2^{t_2-s_2} - i r_2} x^{r_1 2^{s-t} p_1^{s_1-t_1} p_3^{s_3-t_3} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \right) \right\}$$

(5) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then each irreducible polynomial $x^{2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{t_1} p_2^{t_2} p_3^{s_3}} \eta^{k p_3^{t_3-s_3}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{t_1} p_2^{t_2} p_3^{s_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{v_3+s_3}} \left\{ \sum_{r_1=0}^{2^t p_1^{t_1} p_2^{t_2} p_3^{s_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(\zeta^{-j r_1}_{2^t p_1^{t_1} p_2^{t_2} p_3^{s_3}} \eta^{-r_1 k p_3^{t_3-s_3} - i r_2} x^{r_1 2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \right) \right\}$$

(6) If $t \geq s$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then each irreducible polynomial $x^{p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{s_1} p_2^{t_2} p_3^{t_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{s_1} p_2^{t_2} p_3^{t_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{v_3+t_3}} \left\{ \sum_{r_1=0}^{2^s p_1^{s_1} p_2^{t_2} p_3^{t_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(\zeta^{-j r_1}_{2^s p_1^{s_1} p_2^{t_2} p_3^{t_3}} \eta^{-r_1 k 2^{t-s} p_1^{t_1-s_1} - i r_2} x^{r_1 p_2^{s_2-t_2} p_3^{s_3-t_3} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \right) \right\}$$

(7) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then each irreducible polynomial $x^{p_1^{s_1-t_1} p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{t_1} p_2^{s_2} p_3^{t_3}} \eta^{k 2^{t-s} p_2^{t_2-s_2}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{t_1} p_2^{s_2} p_3^{t_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{v_3+t_3}} \left\{ \sum_{r_1=0}^{2^s p_1^{t_1} p_2^{s_2} p_3^{t_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(\zeta^{-j r_1}_{2^s p_1^{t_1} p_2^{s_2} p_3^{t_3}} \eta^{-r_1 k 2^{t-s} p_2^{t_2-s_2} - i r_2} x^{r_1 p_1^{s_1-t_1} p_3^{s_3-t_3} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \right) \right\}$$

(8) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then each irreducible polynomial $x^{p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{t_1} p_2^{t_2} p_3^{s_3}} \eta^{k 2^{t-s} p_3^{t_3-s_3}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{t_1} p_2^{t_2} p_3^{s_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{v_3+s_3}} \left\{ \sum_{r_1=0}^{2^s p_1^{t_1} p_2^{t_2} p_3^{s_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(\zeta^{-j r_1}_{2^s p_1^{t_1} p_2^{t_2} p_3^{s_3}} \eta^{-r_1 k 2^{t-s} p_3^{t_3-s_3} - i r_2} x^{r_1 p_1^{s_1-t_1} p_2^{s_2-t_2} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \right) \right\}$$

(9) If $t < s < v$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then each irreducible polynomial $x^{2^{s-t} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{s_1} p_2^{s_2} p_3^{t_3}} \eta^{k p_1^{t_1-s_1} p_2^{t_2-s_2}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{s_1} p_2^{s_2} p_3^{t_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+t_3}} \left\{ \sum_{r_1=0}^{2^t p_1^{s_1} p_2^{s_2} p_3^{t_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \left(\zeta^{-j r_1}_{2^t p_1^{s_1} p_2^{s_2} p_3^{t_3}} \eta^{-r_1 k p_1^{t_1-s_1} p_2^{t_2-s_2} - i r_2} \right. \right.$$

$$x^{r_1 2^{s-t} p_3^{s_3-t_3} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}}\Big)\Big\}$$

(10) If $t < s < v$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then each irreducible polynomial $x^{2^{s-t} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{s_1} p_2^{t_2} p_3^{s_3}} \eta^{k p_1^{t_1-s_1} p_3^{t_3-s_3}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{s_1} p_2^{t_2} p_3^{s_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{v_3+s_3}} \Bigg\{ \sum_{r_1=0}^{2^t p_1^{s_1} p_2^{t_2} p_3^{s_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \Big(\zeta^{-j r_1}_{2^t p_1^{s_1} p_2^{t_2} p_3^{s_3}} \eta^{-r_1 k p_1^{t_1-s_1} p_3^{t_3-s_3} - i r_2} x^{r_1 2^{s-t} p_2^{s_2-t_2} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \Big) \Bigg\}$$

(11) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then each irreducible polynomial $x^{2^{s-t} p_1^{s_1-t_1}} - \zeta^j_{2^t p_1^{t_1} p_2^{s_2} p_3^{s_3}} \eta^{k p_2^{t_2-s_2} p_3^{t_3-s_3}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{t_1} p_2^{s_2} p_3^{s_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{v_3+s_3}} \Bigg\{ \sum_{r_1=0}^{2^t p_1^{t_1} p_2^{s_2} p_3^{s_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \Big(\zeta^{-j r_1}_{2^t p_1^{t_1} p_2^{s_2} p_3^{s_3}} \eta^{-r_1 k p_2^{t_2-s_2} p_3^{t_3-s_3} - i r_2} x^{r_1 2^{s-t} p_1^{s_1-t_1} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \Big) \Bigg\}$$

(12) If $t \geq s$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then each irreducible polynomial $x^{p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{s_1} p_2^{s_2} p_3^{t_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{s_1} p_2^{s_2} p_3^{t_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{v_3+t_3}} \Bigg\{ \sum_{r_1=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{t_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \Big(\zeta^{-j r_1}_{2^s p_1^{s_1} p_2^{s_2} p_3^{t_3}} \eta^{-r_1 k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2} - i r_2} x^{r_1 p_3^{s_3-t_3} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \Big) \Bigg\}$$

(13) If $t \geq s$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then each irreducible polynomial $x^{p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{s_1} p_2^{t_2} p_3^{s_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_3^{t_3-s_3}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{s_1} p_2^{t_2} p_3^{s_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{v_3+s_3}} \Bigg\{ \sum_{r_1=0}^{2^s p_1^{s_1} p_2^{t_2} p_3^{s_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \Big(\zeta^{-j r_1}_{2^s p_1^{s_1} p_2^{t_2} p_3^{s_3}} \eta^{-r_1 k 2^{t-s} p_1^{t_1-s_1} p_3^{t_3-s_3} - i r_2} x^{r_1 p_2^{s_2-t_2} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \Big) \Bigg\}$$

(14) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then each irreducible polynomial $x^{p_1^{s_1-t_1}} - \zeta^j_{2^s p_1^{t_1} p_2^{s_2} p_3^{s_3}} \eta^{k 2^{t-s} p_2^{t_2-s_2} p_3^{t_3-s_3}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s p_1^{t_1} p_2^{s_2} p_3^{s_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{v_3+s_3}} \Bigg\{ \sum_{r_1=0}^{2^s p_1^{t_1} p_2^{s_2} p_3^{s_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \Big(\zeta^{-j r_1}_{2^s p_1^{t_1} p_2^{s_2} p_3^{s_3}} \eta^{-r_1 k 2^{t-s} p_2^{t_2-s_2} p_3^{t_3-s_3} - i r_2} x^{r_1 p_1^{s_1-t_1} + r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \Big) \Bigg\}$$

(15) If $t < s < v$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then each irreducible polynomial $x^{2^{s-t}} - \zeta^j_{2^t p_1^{s_1} p_2^{s_2} p_3^{s_3}} \eta^{k p_1^{t_1-s_1} p_2^{t_2-s_2} p_3^{t_3-s_3}}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t p_1^{s_1} p_2^{s_2} p_3^{s_3} - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}} \Bigg\{ \sum_{r_1=0}^{2^t p_1^{s_1} p_2^{s_2} p_3^{s_3}-1} \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1} \Big(\zeta^{-j r_1}_{2^t p_1^{s_1} p_2^{s_2} p_3^{s_3}} \eta^{-r_1 k p_1^{t_1-s_1} p_2^{t_2-s_2} p_3^{t_3-s_3} - i r_2}$$

$$x^{r_1 2^{s-t}+r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}}\Big)\Big\}$$

(16) If $t \ge s$, $t_1 \ge s_1$, $t_2 \ge s_2$ and $t_3 \ge s_3$, then each irreducible polynomial $x-\zeta^{j}_{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}}\eta^{k2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2} p_3^{t_3-s_3}}$ over $\mathbb{F}_l$ for $0 \le j \le 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}-1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}}\left\{\sum_{r_1=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}-1}\ \sum_{r_2=0}^{2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}-1}\left(\zeta^{-jr_1}_{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}}\eta^{-r_1 k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2} p_3^{t_3-s_3}-ir_2}\, x^{r_1+r_2 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}}\right)\right\}$$

Theorem 4.8: In Theorem 3.7, suppose that $i=\prod_{i'}\left(p_{i'}^{t_{i'}}\right)k$, $\gcd(k,\prod_{i'} p_{i'})$ =1. Then all the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^s\prod_{i'}\left(p_{i'}^{v_{i'}+s_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}-1\rangle$ are given as follows:

When $t_{i'}=0$ for all i', each irreducible polynomial $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}-\eta^i$ over $\mathbb{F}_l$ corresponds to the primitive idempotent:

$$\theta_i(x) = \frac{1}{\prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}\sum_{r=0}^{\prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1}\eta^{-ri}x^{r\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}.$$

When $t_{i'} \ne 0$ for all i', we divide into the following cases:

(1) If $t_{i'} < s_{i'} < v_{i'}$ for all i', then each irreducible polynomial $x^{\prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)}-\zeta^{j}_{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}\eta^k$ over $\mathbb{F}_l$ for $0 \le j \le \prod_{i'}\left(p_{i'}^{t_{i'}}\right)-1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{\prod_{i'}\left(p_{i'}^{v_{i'}+t_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}\left\{\sum_{r_1=0}^{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)-1}\ \sum_{r_2=0}^{\prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1}\left(\zeta^{-jr_1}_{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}\eta^{-r_1k-ir_2}\, x^{r_1\prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)+r_2\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}\right)\right\}.$$

(2) If $t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \ge s_{v'}$, $1 \le u', v' \le r$, $u' \ne v' \ne j'$. then each irreducible polynomial $x^{\prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)}-\zeta^{j}_{\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)}\eta^{k\prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}$ over $\mathbb{F}_l$ for $0 \le j \le \prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)-1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{\prod_{u'}\left(p_{u'}^{v_{u'}+t_{u'}}\right)\prod_{v'}\left(p_{v'}^{v_{v'}+s_{v'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}\left\{\sum_{r_1=0}^{\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)-1}\ \sum_{r_2=0}^{\prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1}\left(\zeta^{-jr_1}_{\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)}\eta^{-r_1k\prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)-ir_2}\, x^{r_1\prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)+r_2\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}\right)\right\}.$$

(3) If $t_{i'} \ge s_{i'}$ for all i', then each irreducible polynomial $x-\zeta^{j}_{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}\eta^{k\prod_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)}$ over $\mathbb{F}_l$ for $0 \le j \le \prod_{i'}\left(p_{i'}^{s_{i'}}\right)-1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{\prod_{i'}\left(p_{i'}^{v_{i'}+s_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}\left\{\sum_{r_1=0}^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)-1}\sum_{r_2=0}^{\prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1}\left(\zeta_{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}^{-jr_1}\eta^{-r_1 k\prod_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)-ir_2} x^{r_1+r_2\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}\right)\right\}.$$

Theorem 4.9: In Theorem 3.8, suppose that $i = 2^t\prod_{i'}\left(p_{i'}^{t_{i'}}\right)k$, $\gcd(k, 2\prod_{i'} p_{i'})$ =1. Then all the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^{v+s}\prod_{i'}\left(p_{i'}^{v_{i'}+s_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)} - 1\rangle$ are given as follows:

When $(t, t_{i'}) = 0$ for all i', then each irreducible polynomial $x^{2^s\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$ over $\mathbb{F}_l$ corresponds to the primitive idempotent:

$$\theta_i(x) = \frac{1}{2^v\prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}\sum_{r=0}^{2^v\prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1}\eta^{-ri}x^{r2^s\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}.$$

When $(t, t_{i'}) \neq 0$ for all i',we divide into the following cases:

(1) If $t < s < v,\ t_{i'} < s_{i'} < v_{i'}$ for all i', then each irreducible polynomial $x^{2^{s-t}\prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)} - \zeta_{2^t\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}^{j}\eta^k$ over $\mathbb{F}_l$ for $0\leq j \leq 2^t\prod_{i'}\left(p_{i'}^{t_{i'}}\right) - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t}\prod_{i'}\left(p_{i'}^{v_{i'}+t_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}\left\{\sum_{r_1=0}^{2^t\prod_{i'}\left(p_{i'}^{t_{i'}}\right)-1}\sum_{r_2=0}^{2^v\prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1}\left(\zeta_{2^t\prod_{i'}\left(p_{i'}^{t_{i'}}\right)}^{-jr_1}\eta^{-r_1 k-ir_2} x^{r_1 2^{s-t}\prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)+r_2 2^s\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}\right)\right\}.$$

(2) If $t \geq s,\ t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \geq s_{v'},\ 1 \leq u', v' \leq e,\ u' \neq v' \neq j'$, then each irreducible polynomial $x^{\prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)} - \zeta_{2^s\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)}^{j}\eta^{k2^{t-s}\prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^s\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right) - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s}\prod_{u'}\left(p_{u'}^{v_{u'}+t_{u'}}\right)\prod_{v'}\left(p_{v'}^{v_{v'}+s_{v'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}\left\{\sum_{r_1=0}^{2^s\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)-1}\sum_{r_2=0}^{2^v\prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1}\left(\zeta_{2^s\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)}^{-jr_1}\eta^{-r_1 k2^{t-s}\prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)-ir_2} x^{r_1\prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)+r_2 2^s\prod_{i'}\left(p_{i'}^{s_{i'}}\right)}\right)\right\}.$$

(3) If $t < s < v,\ t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \geq s_{v'},\ 1 \leq u', v' \leq e,\ u' \neq v' \neq j'$, then each irreducible polynomial $x^{2^{s-t}\prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)} - \zeta_{2^s\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)}^{j}\eta^{k\prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}$ over $\mathbb{F}_l$ for $0 \leq j \leq 2^t\prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right) - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+t}\prod_{u'}\left(p_{u'}^{v_{u'}+t_{u'}}\right)\prod_{v'}\left(p_{v'}^{v_{v'}+s_{v'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)}$$

$$\left\{ \sum_{r_1=0}^{2^t \prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)-1} \sum_{r_2=0}^{2^v \prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1} \left(\zeta_{2^t \prod_{u'}\left(p_{u'}^{t_{u'}}\right)\prod_{v'}\left(p_{v'}^{s_{v'}}\right)}^{-jr_1} \eta^{-r_1 k \prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)-ir_2} x^{r_1 2^{s-t}\prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)+r_2 2^s \prod_{i'}\left(p_{i'}^{s_{i'}}\right)} \right) \right\}.$$

(4) If $t \ge s$, $t_{i'} \ge s_{i'}$ for all i', then each irreducible polynomial $x - \zeta_{2^s \prod_{i'}\left(p_{i'}^{s_{i'}}\right)}^{j} \eta^{k 2^{t-s}\prod_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)}$ over $\mathbb{F}_l$ for $0 \le j \le 2^s \prod_{i'}\left(p_{i'}^{s_{i'}}\right) - 1$ corresponds to the primitive idempotent:

$$\theta_{i,j}(x) = \frac{1}{2^{v+s}\prod_{i'}\left(p_{i'}^{v_{i'}+s_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)} \left\{ \sum_{r_1=0}^{2^s \prod_{i'}\left(p_{i'}^{s_{i'}}\right)-1} \sum_{r_2=0}^{2^v \prod_{i'}\left(p_{i'}^{v_{i'}}\right)\prod_{j'}\left(p_{j'}^{s_{j'}}\right)-1} \left(\zeta_{2^s \prod_{i'}\left(p_{i'}^{s_{i'}}\right)}^{-jr_1} \eta^{-r_1 k 2^{t-s}\prod_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)-ir_2} x^{r_1 + r_2 2^s \prod_{i'}\left(p_{i'}^{s_{i'}}\right)} \right) \right\}.$$

V. THE MINIMUM HAMMING DISTANCES OF THE IRREDUCIBLE CYCLIC CODES

In this section, first we give the check polynomials, minimum Hamming distances and the dimensions of the irreducible cyclic codes generated by the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}} - 1\rangle$ for $e = 3$ and then give generalized result. We indicate that the parameters of the irreducible cyclic codes in $\mathbb{F}_l[x]/\langle x^{2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}} - 1\rangle$ also can be easily obtained based on the primitive idempotents in $\mathbb{F}_l[x]/\langle x^{2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}} - 1\rangle$.

Let $\mathcal{C}$ denote an irreducible cyclic code of length $2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}$ generated by a primitive idempotent $\theta(x)$, whose check polynomial is an irreducible divisor of $x^{2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}} - 1$. It is clear that $\mathcal{C} = \langle \theta(x) \rangle = \langle g(x) \rangle$, where $g(x) = \gcd\left(\theta(x), x^{2^a p_1^{a_1} p_2^{a_2} \dots p_e^{a_e}} - 1\right)$ is called the generator polynomial of the irreducible cyclic code $\mathcal{C}$.

Theorem 5.1: In Theorem 4.2, suppose that $i = p_3^{t_3} k$, $\gcd(k, p_3) = 1$, $0 \le i \le 2^s p_1^{s_1} p_2^{s_2} p_3^{v_3} - 1$.

(1) If $t_3 = 0$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $\left[2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}, p_3^{s_3}, 2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}\right]$ cyclic code whose check polynomial is $x^{p_3^{s_3}} - \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{i}$.

(2) If $t_3 < s_3$, $\mathcal{C}_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le p_3^{t_3} - 1$, is an $\left[2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}, p_3^{s_3-t_3}, 2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+t_3}\right]$ cyclic code whose check polynomial is $x^{p_3^{s_3-t_3}} - \zeta_{p_3^{t_3}}^{j} \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{k}$.

(3) If $t_3 \ge s_3$, $\mathcal{C}_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le p_3^{s_3} - 1$, is an $\left[2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}, 1, 2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}\right]$ cyclic code whose check polynomial is $x - \zeta_{p_3^{s_3}}^{j} \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{k p_3^{t_3-s_3}}$.

Proof: If $t_3 = 0$, then by the construction of $\theta_i(x)$, we have a ring isomorphism:

$\mathcal{C}_i = \langle \theta_i(x) \rangle = \theta_i(x) R_n \cong \mathbb{F}_l[x]/\langle x^{p_3^{s_3}} - \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{i} \rangle$,

where $R_n = \mathbb{F}_l[x]/\langle x^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} - 1\rangle$. Hence $x^{p_3^{s_3}} - \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{i}$ is the check polynomial of $\mathcal{C}_i$. On the other hand, if $c(x) \in \mathbb{F}_l[x]$, then by division algorithm $c(x) = q(x)\left(x^{p_3^{s_3}} - \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{i}\right) + r(x)$, where $q(x), r(x) \in \mathbb{F}_l[x]$, $\deg r(x) < p_3^{s_3}$. Then we have

$c(x)\theta_i(x) \equiv r(x)\theta_i(x) \left(\bmod\ x^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+s_3}} - 1\right)$.

Recall that

$$\theta_i(x) = \frac{1}{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}} \sum_{r=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}-1} \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{-ri} x^{rp_3^{s_3}}.$$

The degree of each item about $\theta_i(x)$ differs at least $p_3^{s_3}$ and $\deg r(x) < p_3^{s_3}$. If $r(x) \neq 0$, then we get the Hamming weight of $\overline{c(x)\theta_i(x)}$ in R_n is just

$W_H(r(x)\theta_i(x)) = W_H(r(x)).W_H(\theta_i(x)) \geq W_H(\theta_i(x)) = 2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}$.

In fact, the equality holds if and only if $W_H(r(x)) = 1$. Thus we have $d_H(\mathcal{C}_i) = 2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}$.

If $t_3 < s_3$, then by the construction of $\theta_{i,j}(x)$, we have a ring isomorphism:

$$\mathcal{C}_{i,j} = \langle \theta_{i,j}(x) \rangle = R_n \theta_{i,j}(x) \cong \mathbb{F}_l[x]/\langle x^{p_3^{s_3-t_3}} - \zeta_{p_3^{t_3}}^{j} \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{k} \rangle.$$

Hence $x^{p_3^{s_3-t_3}} - \zeta_{p_3^{t_3}}^{j} \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{k}$ is the check polynomial of $\mathcal{C}_{i,j}$.

Recall that

$$\theta_{i,j}(x) = \frac{1}{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+t_3}} \sum_{r_1=0}^{p_3^{t_3}-1} \sum_{r_2=0}^{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}-1} \zeta_{p_3^{t_3}}^{-jr_1} \zeta_{2^s p_1^{s_1} p_2^{s_2} p_3^{v_3}}^{-r_1 k - ir_2} x^{r_1 p_3^{s_3-t_3} + r_2 p_3^{s_3}}.$$

Similarly, we have $d_H(\mathcal{C}_{i,j}) = 2^s p_1^{s_1} p_2^{s_2} p_3^{v_3+t_3}$.

We can also get the proof in the case $t_3 \geq s_3$.

By the same method, we can get the following theorems:

Theorem 5.2: In Theorem 4.4, suppose that $i = 2^t k$, gcd$(k, 2)$ =1, $0 \leq i \leq 2^v p_1^{s_1} p_2^{s_2} p_3^{s_3} - 1$.

(1) If $t_1 = 0$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{s_1} p_2^{s_2} p_3^{s_3}, 2^s, 2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^s} - \zeta_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}^{i}$.

(2) If $t_1 < s_1$, $\mathcal{C}_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq 2^t - 1$, is an $[2^{v+s} p_1^{s_1} p_2^{s_2} p_3^{s_3}, 2^{s-t}, 2^{v+t} p_1^{s_1} p_2^{s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t}} - \zeta_{2^t}^{j} \zeta_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}^{k}$.

(3) If $t_1 \geq s_1$, $\mathcal{C}_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq 2^s - 1$, is an $[2^{v+s} p_1^{s_1} p_2^{s_2} p_3^{s_3}, 1, 2^{v+s} p_1^{s_1} p_2^{s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x - \zeta_{2^s}^{j} \zeta_{2^v p_1^{s_1} p_2^{s_2} p_3^{s_3}}^{k2^{t-s}}$.

Theorem 5.3: In Theorem 4.5, suppose that $i = 2^t p_1^{t_1} k$, $\gcd(k, l_1 l_2) = 1$, $0 \leq i \leq 2^v p_1^{v_1} p_2^{s_2} p_3^{s_3} - 1$. When $(t, t_1) = (0,0)$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}, 2^s p_1^{s_1}, 2^v p_1^{v_1} p_2^{s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^s p_1^{s_1}} - \eta^i$.

When $(t, t_1) \neq (0,0)$, we divide into four cases:

(1) If $t < s < v$ and $t_1 < s_1 < v_1$, then $\mathcal{C}_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq 2^t p_1^{t_1} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}, 2^{s-t} p_1^{s_1-t_1}, 2^{v+t} p_1^{v_1+t_1} p_2^{s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_1^{s_1-t_1}} - \zeta_{2^t p_1^{t_1}}^{j} \eta^k$.

(2) If $t \geq s$ and $t_1 < s_1 < v_1$, then $\mathcal{C}_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq 2^s p_1^{t_1} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}, p_1^{s_1-t_1}, 2^{v+s} p_1^{v_1+t_1} p_2^{s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{p_1^{s_1-t_1}} - \zeta_{2^s p_1^{t_1}}^{j} \eta^{k2^{t-s}}$.

(3) If $t < s < v$ and $t_1 \geq s_1$, then $\mathcal{C}_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq 2^t p_1^{s_1} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}, 2^{s-t}, 2^{v+t} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t}} - \zeta_{2^t p_1^{s_1}}^{j} \eta^{kp_1^{t_1-s_1}}$.

(4) If $t \geq s$ and $t_1 \geq s_1$, then $\mathcal{C}_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq 2^s p_1^{s_1} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}, 1, 2^{v+s} p_1^{v_1+s_1} p_2^{s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x - \zeta_{2^s p_1^{s_1}}^{j} \eta^{k2^{t-s} p_1^{t_1-s_1}}$.

Theorem 5.4: In Theorem 4.6, suppose that $i = 2^t p_1^{t_1} p_2^{t_2} k$, $\gcd(k, 2p_1p_2) = 1$, $0 \le i \le 2^v p_1^{v_1} p_2^{v_2} p_3^{s_3} - 1$. When $(t, t_1, t_2) = (0,0,0)$, then $C_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3},\ 2^s p_1^{s_1} p_2^{s_2},\ 2^v p_1^{v_1} p_2^{v_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^s p_1^{s_1} p_2^{s_2}} - \eta^i$.

When $(t, t_1, t_2) \ne (0,0,0)$, we divide into eight cases:

(1) If $t < s < v$, $t_1 < s_1 < v_1$ and $t_2 < s_2 < v_2$, then $C_i = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le 2^t p_1^{t_1} p_2^{t_2} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3},\ 2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2},\ 2^{v+t} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{t_1} p_2^{t_2}} \eta^k$.

(2) If $t \ge s$, $t_1 < s_1 < v_1$ and $t_2 < s_2 < v_2$, then $C_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le 2^s p_1^{t_1} p_2^{t_2} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3},\ p_1^{s_1-t_1} p_2^{s_2-t_2},\ 2^{v+s} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{t_1} p_2^{t_2}} \eta^{k2^{t-s}}$.

(3) If $t < s < v$, $t_1 \ge s_1$ and $t_2 < s_2 < v_2$, then $C_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le 2^t p_1^{s_1} p_2^{t_2} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3},\ 2^{s-t} p_2^{s_2-t_2},\ 2^{v+t} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{s_1} p_2^{t_2}} \eta^{k p_1^{t_1-s_1}}$.

(4) If $t < s < v$, $t_1 < s_1 < v_1$ and $t_2 \ge s_2$, then $C_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le 2^t p_1^{t_1} p_2^{s_2} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3},\ 2^{s-t} p_1^{s_1-t_1},\ 2^{v+t} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_1^{s_1-t_1}} - \zeta^j_{2^t p_1^{t_1} p_2^{s_2}} \eta^{k p_2^{t_2-s_2}}$.

(5) If $t < s < v$, $t_1 \ge s_1$ and $t_2 \ge s_2$, then $C_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le 2^t p_1^{s_1} p_2^{s_2} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3},\ 2^{s-t},\ 2^{v+t} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t}} - \zeta^j_{2^t p_1^{s_1} p_2^{s_2}} \eta^{k p_1^{t_1-s_1} p_2^{t_2-s_2}}$.

(6) If $t \ge s$, $t_1 < s_1 < v_1$ and $t_2 \ge s_2$, then $C_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le 2^s p_1^{t_1} p_2^{s_2} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3},\ p_1^{s_1-t_1},\ 2^{v+s} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{p_1^{s_1-t_1}} - \zeta^j_{2^s p_1^{t_1} p_2^{s_2}} \eta^{k 2^{t-s} p_2^{t_2-s_2}}$.

(7) If $t \ge s$, $t_1 \ge s_1$ and $t_2 < s_2 < v_2$, then $C_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le 2^s p_1^{s_1} p_2^{t_2} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3},\ p_2^{s_2-t_2},\ 2^{v+s} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x^{p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{s_1} p_2^{t_2}} \eta^{k 2^{t-s} p_1^{t_1-s_1}}$.

(8) If $t \ge s$, $t_1 \ge s_1$ and $t_2 \ge s_2$, then $C_{i,j} = \langle \theta_{i,j}(x) \rangle$, $0 \le j \le 2^s p_1^{s_1} p_2^{s_2} - 1$, is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3},\ 1,\ 2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{s_3}]$ cyclic code whose check polynomial is $x - \zeta^j_{2^s p_1^{s_1} p_2^{s_2}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2}}$.

Theorem 5.5: In Theorem 4.7, suppose that $i = 2^t p_1^{t_1} p_2^{t_2} p_3^{t_3} k$, $\gcd(k, 2p_1p_2p_3) = 1$, $0 \le i \le 2^v p_1^{v_1} p_2^{v_2} p_3^{v_3+s_3} - 1$. When $(t, t_1, t_2, t_3) = (0,0,0,0)$, then $C_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3},\ 2^s p_1^{s_1} p_2^{s_2} p_3^{s_3},\ 2^v p_1^{v_1} p_2^{v_2} p_3^{v_3}]$ cyclic code whose check polynomial is $x^{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} - \eta^i$.

When $(t, t_1, t_2, t_3) \ne (0,0,0,0)$, we divide into sixteen cases:

(1) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then $C_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3},\ 2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3},\ 2^{v+t} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{v_3+t_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{t_1} p_2^{t_2} p_3^{t_3}} \eta^k$.

(2) If $t \ge s$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then $C_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3}, 2^{v+s} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{v_3+t_3}]$ cyclic code whose check polynomial is $x^{p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{t_1} p_2^{t_2} p_3^{t_3}} \eta^{k2^{t-s}}$.

(3) If $t < s < v$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, 2^{s-t} p_2^{s_2-t_2} p_3^{s_3-t_3}, 2^{v+t} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{v_3+t_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{s_1} p_2^{t_2} p_3^{t_3}} \eta^{k p_1^{t_1-s_1}}$.

(4) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, 2^{s-t} p_1^{s_1-t_1} p_3^{s_3-t_3}, 2^{v+t} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{v_3+t_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_1^{s_1-t_1} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{t_1} p_2^{s_2} p_3^{t_3}} \eta^{k p_2^{t_2-s_2}}$.

(5) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, 2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2}, 2^{v+t} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{v_3+s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{t_1} p_2^{t_2} p_3^{s_3}} \eta^{k p_3^{t_3-s_3}}$.

(6) If $t \geq s$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 < s_3 < v_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, p_2^{s_2-t_2} p_3^{s_3-t_3}, 2^{v+s} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{v_3+t_3}]$ cyclic code whose check polynomial is $x^{p_2^{s_2-t_2} p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{s_1} p_2^{t_2} p_3^{t_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1}}$.

(7) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, p_1^{s_1-t_1} p_3^{s_3-t_3}, 2^{v+s} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{v_3+t_3}]$ cyclic code whose check polynomial is $x^{p_1^{s_1-t_1} p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{t_1} p_2^{s_2} p_3^{t_3}} \eta^{k 2^{t-s} p_2^{t_2-s_2}}$.

(8) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, p_1^{s_1-t_1} p_2^{s_2-t_2}, 2^{v+s} p_1^{v_1+t_1} p_2^{v_2+t_2} p_3^{v_3+s_3}]$ cyclic code whose check polynomial is $x^{p_1^{s_1-t_1} p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{t_1} p_2^{t_2} p_3^{s_3}} \eta^{k 2^{t-s} p_3^{t_3-s_3}}$.

(9) If $t < s < v$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, 2^{s-t} p_3^{s_3-t_3}, 2^{v+t} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+t_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_3^{s_3-t_3}} - \zeta^j_{2^t p_1^{s_1} p_2^{s_2} p_3^{t_3}} \eta^{k p_1^{t_1-s_1} p_2^{t_2-s_2}}$.

(10) If $t < s < v$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, 2^{s-t} p_2^{s_2-t_2}, 2^{v+t} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{v_3+s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_2^{s_2-t_2}} - \zeta^j_{2^t p_1^{s_1} p_2^{t_2} p_3^{s_3}} \eta^{k p_1^{t_1-s_1} p_3^{t_3-s_3}}$.

(11) If $t < s < v$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, 2^{s-t} p_1^{s_1-t_1}, 2^{v+t} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{v_3+s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t} p_1^{s_1-t_1}} - \zeta^j_{2^t p_1^{t_1} p_2^{s_2} p_3^{s_3}} \eta^{k p_2^{t_2-s_2} p_3^{t_3-s_3}}$.

(12) If $t \geq s$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 < s_3 < v_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, p_3^{s_3-t_3}, 2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+t_3}]$ cyclic code whose check polynomial is $x^{p_3^{s_3-t_3}} - \zeta^j_{2^s p_1^{s_1} p_2^{s_2} p_3^{t_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2}}$.

(13) If $t \geq s$, $t_1 \geq s_1$, $t_2 < s_2 < v_2$ and $t_3 \geq s_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, p_2^{s_2-t_2}, 2^{v+s} p_1^{v_1+s_1} p_2^{v_2+t_2} p_3^{v_3+s_3}]$ cyclic code whose check polynomial is $x^{p_2^{s_2-t_2}} - \zeta^j_{2^s p_1^{s_1} p_2^{t_2} p_3^{s_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_3^{t_3-s_3}}$.

(14) If $t \geq s$, $t_1 < s_1 < v_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, p_1^{s_1-t_1}, 2^{v+s} p_1^{v_1+t_1} p_2^{v_2+s_2} p_3^{v_3+s_3}]$ cyclic code whose check polynomial is $x^{p_1^{s_1-t_1}} - \zeta^j_{2^s p_1^{t_1} p_2^{s_2} p_3^{s_3}} \eta^{k 2^{t-s} p_2^{t_2-s_2} p_3^{t_3-s_3}}$.

(15) If $t < s < v$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, 2^{s-t}, 2^{v+t} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}]$ cyclic code whose check polynomial is $x^{2^{s-t}} - \zeta^j_{2^t p_1^{s_1} p_2^{s_2} p_3^{s_3}} \eta^{k p_1^{t_1-s_1} p_2^{t_2-s_2} p_3^{t_3-s_3}}$.

(16) If $t \geq s$, $t_1 \geq s_1$, $t_2 \geq s_2$ and $t_3 \geq s_3$, then $\mathcal{C}_i = \langle \theta_i(x) \rangle$ is an $[2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}, 1, 2^{v+s} p_1^{v_1+s_1} p_2^{v_2+s_2} p_3^{v_3+s_3}]$ cyclic code whose check polynomial is $x - \zeta^j_{2^s p_1^{s_1} p_2^{s_2} p_3^{s_3}} \eta^{k 2^{t-s} p_1^{t_1-s_1} p_2^{t_2-s_2} p_3^{t_3-s_3}}$.

Now we give the following theorems as generalization of above results:

Theorem 5.6: In Theorem 4.8, suppose that $i = \prod_{i'} \left(p_{i'}^{t_{i'}}\right) k$, $\gcd(k, \prod_{i'} p_{i'}) = 1$, $0 \leq i \leq \prod_{i'} \left(p_{i'}^{v_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right) - 1$. When $t_{i'} = 0$ for all i', then $\mathcal{C}_i = \langle \theta_i(x) \rangle$, is an $\left[\prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right), \prod_{i'} \left(p_{i'}^{s_{i'}}\right), \prod_{i'} \left(p_{i'}^{v_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right)\right]$ cyclic code whose check polynomial is $x^{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$.

When $t_{i'} \neq 0$ for all i', we divide into the following cases:

(1) If $t_{i'} < s_{i'} < v_{i'}$ for all i', then $\mathcal{C}_i = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq \prod_{i'} \left(p_{i'}^{t_{i'}}\right) - 1$, is an $\left[\prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right), \prod_{i'} \left(p_{i'}^{s_{i'}-t_{i'}}\right), \prod_{i'} \left(p_{i'}^{v_{i'}+t_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right)\right]$ cyclic code whose check polynomial is $x^{\prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)} - \zeta^j_{\prod_{i'}\left(p_{i'}^{t_{i'}}\right)} \eta^k$.

(2) If $t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \geq s_{v'}$, $1 \leq u', v' \leq r$, $u' \neq v' \neq j'$, then $\mathcal{C}_i = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq \prod_{u'} \left(p_{u'}^{t_{u'}}\right) \prod_{v'} \left(p_{v'}^{s_{v'}}\right) - 1$ is an $\left[\prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right), \prod_{i'} \left(p_{i'}^{s_{u'}-t_{u'}}\right), \prod_{i'} \left(p_{u'}^{v_{u'}+t_{u'}}\right) \prod_{i'} \left(p_{v'}^{v_{v'}+s_{v'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right)\right]$ cyclic code whose check polynomial is $x^{\prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)} - \zeta^j_{\prod_{u'}\left(p_{u'}^{t_{u'}}\right) \prod_{v'}\left(p_{v'}^{s_{v'}}\right)} \eta^{k \prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}$.

(3) If $t_{i'} \geq s_{i'}$ for all i', then $\mathcal{C}_i = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq \prod_{i'} \left(p_{i'}^{s_{i'}}\right) - 1$ is an $\left[\prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right), 1, \prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right)\right]$ cyclic code whose check polynomial is $x - \zeta^j_{\prod_{i'}\left(p_{i'}^{s_{i'}}\right)} \eta^{k \prod_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)}$.

Theorem 5.7: In Theorem 4.9, suppose that $i = 2^t \prod_{i'} \left(p_{i'}^{t_{i'}}\right) k$, $\gcd(k, \prod_{i'} p_{i'}) = 1$, $0 \leq i \leq 2^v \prod_{i'} \left(p_{i'}^{v_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right) - 1$. When $(t, t_{i'}) = 0$ for all i', then $\mathcal{C}_i = \langle \theta_i(x) \rangle$, is an $\left[2^{v+s} \prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right), 2^s \prod_{i'} \left(p_{i'}^{s_{i'}}\right), 2^v \prod_{i'} \left(p_{i'}^{v_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right)\right]$ cyclic code whose check polynomial is $x^{2^s \prod_{i'}\left(p_{i'}^{s_{i'}}\right)} - \eta^i$.

When $(t, t_{i'}) \neq 0$ for all i', we divide into the following cases:

(1) If $t < s < v$, $t_{i'} < s_{i'} < v_{i'}$ for all i', then $\mathcal{C}_i = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq 2^t \prod_{i'} \left(p_{i'}^{t_{i'}}\right) - 1$, is an $\left[2^{v+s} \prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right), 2^{s-t} \prod_{i'} \left(p_{i'}^{s_{i'}-t_{i'}}\right), 2^{v+t} \prod_{i'} \left(p_{i'}^{v_{i'}+t_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right)\right]$ cyclic code whose check polynomial is $x^{2^{s-t} \prod_{i'}\left(p_{i'}^{s_{i'}-t_{i'}}\right)} - \zeta^j_{2^t \prod_{i'}\left(p_{i'}^{t_{i'}}\right)} \eta^k$.

(2) If $t \geq s$, $t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \geq s_{v'}$, $1 \leq u', v' \leq e$, $u' \neq v' \neq j'$, then $\mathcal{C}_i = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq 2^s \prod_{u'} \left(p_{u'}^{t_{u'}}\right) \prod_{v'} \left(p_{v'}^{s_{v'}}\right) - 1$ is an $\left[2^{v+s} \prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right), \prod_{u'} \left(p_{u'}^{s_{u'}-t_{u'}}\right), 2^{v+s} \prod_{i'} \left(p_{u'}^{v_{u'}+t_{u'}}\right) \prod_{i'} \left(p_{v'}^{v_{v'}+s_{v'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right)\right]$ cyclic code whose check polynomial is $x^{\prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)} - \zeta^j_{2^s \prod_{u'}\left(p_{u'}^{t_{u'}}\right) \prod_{v'}\left(p_{v'}^{s_{v'}}\right)} \eta^{k 2^{t-s} \prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}$.

(3) If $t < s < v$, $t_{u'} < s_{u'} < v_{u'}$ and $t_{v'} \geq s_{v'}, 1 \leq u', v' \leq e$, $u' \neq v' \neq j'$, then $\mathcal{C}_i = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq 2^t \prod_{u'} \left(p_{u'}^{t_{u'}}\right) \prod_{v'} \left(p_{v'}^{s_{v'}}\right) - 1$ is an $\left[2^{v+s} \prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right),\ 2^{s-t} \prod_{u'} \left(p_{u'}^{s_{u'}-t_{u'}}\right),\ 2^{v+t} \prod_{i'} \left(p_{u'}^{v_{u'}+t_{u'}}\right) \prod_{i'} \left(p_{v'}^{v_{v'}+s_{v'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right)\right]$ cyclic code whose check polynomial is $x^{2^{s-t} \prod_{u'}\left(p_{u'}^{s_{u'}-t_{u'}}\right)} - \zeta^{j}_{2^s \prod_{u'}\left(p_{u'}^{t_{u'}}\right) \prod_{v'}\left(p_{v'}^{s_{v'}}\right)} \eta^{k \prod_{v'}\left(p_{v'}^{t_{v'}-s_{v'}}\right)}$.

(4) If $t \geq s$, $t_{i'} \geq s_{i'}$ for all i', then $\mathcal{C}_i = \langle \theta_{i,j}(x) \rangle$, $0 \leq j \leq \prod_{i'} \left(p_{i'}^{s_{i'}}\right) - 1$ is an $\left[2^{v+s} \prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right),\ 1,\ 2^{v+s} \prod_{i'} \left(p_{i'}^{v_{i'}+s_{i'}}\right) \prod_{j'} \left(p_{j'}^{s_{j'}}\right)\right]$ cyclic code whose check polynomial is $x - \zeta^{j}_{2^s \prod_{i'}\left(p_{i'}^{s_{i'}}\right)} \eta^{k 2^{t-s} \prod_{i'}\left(p_{i'}^{t_{i'}-s_{i'}}\right)}$.

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