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Oscillation Criteria for First Order Nonlinear Neutral Delay Difference Equations with Variable Coefficients

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ABSTRACT: *We discuss the oscillatory behavior of all solutions of first order nonlinear neutral delay difference equations with variable coefficients of the form*

$$\Delta\big[r(n)\big(a(n)x(n)-p(n)x(n-\tau)\big)\big]+q(n)\,x(n-\sigma)=0;\ n\geq n_0 \qquad (*)$$

where $\{r(n)\}, \{a(n)\}$ are sequences of positive real numbers, $\{p(n)\}$ and $\{q(n)\}$ are sequences of nonnegative real numbers, τ and σ are positive integers. Our proved results extend and develop some of the well-known results in the literature. Examples are inserted to demonstrate the confirmation of our new results.

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I. INTRODUCTION

Neutral delay difference equations contain the difference of the unknown sequence both with and without delays. Some new phenomena can appear, hence the theory of neutral delay difference equations is even more complicated than the theory of non-neutral delay difference equations. The idea of neutral delay difference equations is even more difficult that the theory of non-neutral delay difference equations. The oscillation concept is a part of the qualitative theory of this type of equations has been developed quickly in the past few years. To a significant scope, the analysis of neutral delay difference equations is inspired by having many uses in technology and natural science. Few applications of these equations and some variations in the properties of their solutions and the solutions of non-neutral equations can be observed in Agarwal [1], Gyori and Ladas [6].
In this paper, we are concerned with oscillations of solutions of the following first order nonlinear neutral delay difference equations of the form

$$\Delta\big[r(n)\big(a(n)x(n)-p(n)x(n-\tau)\big)\big]+q(n)x(n-\sigma)=0,\quad n\geq n_0\ ; \qquad (1)$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, $\{r(n)\}$, and $\{a(n)\}$are sequences of positive real numbers, $\{p(n)\}$ and $\{q(n)\}$ are sequences of nonnegative real numbers, τ and σ are positive integers.

The following conditions are assumed to be hold; throughout the paper:
(C_1) $\{q(n)\}$ is not identically zero for sufficiently large values of n;
(C_2) There exists constants a_0 and λ such that $0<a(n)\leq a_0$ and $0\leq\frac{p(n)}{a(n)}\leq\lambda<1$;
Observe that when $r(n)\equiv 1,\ and\ a(n)=1$, (1) reduces to the equation

$$\Delta[x(n)-p(n)x(n-\tau)]+q(n)x(n-\sigma)=0,\quad n\geq n_0 \qquad (2)$$

which has been studied by several authors, see [7,8]. In [9], we established sufficient conditions for oscillation of all solutions of

$$\Delta\big[r(n)\big(a(n)x(n)-p(n)x(n-\tau)\big)\big]+q(n)\,x(n-\sigma))=0,\ n\geq n_0 \quad (3)$$

The results obtained in this paper are discrete analogues of some well known results in [10].
A solution$\{x(n)\}$ of (1) is said to be nonoscillatory if the terms $\{x(n)\}$ are either eventually positive or eventually negative. Otherwise, the solution is called oscillates. For the general background on difference equations, one can refer to [1,3-10].
In the continuation, unless otherwise described, when we write a functional inequality we shall assume that it holds for all sufficiently large values of n.

II. SOME LEMMAS

Lemma 2.1. [6] The difference inequality

$$\Delta x(n) + q(n)x(n-\sigma) \le 0; \qquad n \ge n_0 \qquad (4)$$

has an eventually positive solution if and only if the difference equation

$$\Delta x(n) + q(n)x(n-\sigma) = 0; \qquad n \ge n_0 \qquad (5)$$

has an eventually positive solution.

Lemma 2.2. [6] Assume that

$$lim_{n\to\infty} inf \sum_{s=n-\sigma}^{n-1} q(s) > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}. \qquad (6)$$

Then,

(i) the delay difference inequality

$$\Delta x(n) + q(n)x(n-\sigma) \le 0; \qquad n \ge n_0 \qquad (7)$$

cannot have an eventually positive solution;

(ii)the advanced difference inequality

$$\Delta x(n) - q(n)x(n+\sigma) \ge 0; \qquad n \ge n_0 \qquad (8)$$

cannot have an eventually positive solution.

Lemma 2.3. [6] Assume that

$$lim_{n\to\infty} sup \sum_{s=n}^{n+\sigma} q(s) > 1. \qquad (9)$$

Then every solution of (4) is oscillatory.

Lemma 2.4. [10] Assume that $\{r(n)\}$ is nondecreasing sequence of positive real numbers. Let $\{x(n)\}$ be an eventually positive solution of (1). Set

$$z(n) = a(n)x(n) - p(n)x(n-\tau) \quad (10)$$

Then

$$z(n) > 0 \text{ eventually and } \Delta z(n) < 0 \text{ eventually.} \qquad (11)$$

III. MAIN RESULTS

Our objective in this section is to establish the following results.

Theorem 3.1. Assume that $r(n) \equiv 1$. Assume further that either

$$lim_{\ n\to\infty} inf \ \sum_{s=n-\sigma}^{n-1} \frac{q(s)}{a(s-\sigma)} > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \qquad (12)$$

or

$$lim_{n\to\infty} sup \sum_{s=n}^{n+\sigma} \frac{q(s)}{a(s-\sigma)} > 1 \qquad (13)$$

There every solution of (1) is oscillatory.

Proof. For the benefit of obtaining a contradiction, suppose that there is an eventually positive solution $\{x(n)\}$ of (1) and let $\{z(n)\}$ be its associated sequence obtained by (10).

Then by Lemma 2.4, there exists an integer $n_1 \ge n_0$ such that

$$z(n) > 0; \qquad n \ge n_1 \ge n_0. \qquad (14)$$

From (1), with $r(n) \equiv 1$, we have

$$\Delta z(n) = -q(n)x(n-\sigma)$$

$$= -q(n)\left[\frac{1}{a(n-\sigma)}\left(z(n-\sigma) + p(n-\sigma)x(n-\tau-\sigma)\right)\right]$$

$$= \frac{-q(n)}{a(n-\sigma)} z(n-\sigma) - \frac{q(n)p(n-\sigma)}{a(n-\sigma)} x(n-\tau-\sigma)$$

$$= \frac{-q(n)z(n-\sigma)}{a(n-\sigma)} + \frac{p(n-\sigma)q(n)}{a(n-\sigma)}\frac{\Delta z(n-\tau)}{q(n-\tau)}. \qquad (15)$$

Hence

$$\Delta z(n) - \frac{p(n-\sigma)}{a(n-\sigma)}\frac{q(n)}{q(n-\tau)}\Delta z(n-\tau) + \frac{q(n)}{a(n-\sigma)}z(n-\sigma) = 0. \qquad (16)$$

or

$$\Delta z(n) + \frac{q(n)}{a(n-\sigma)}z(n-\sigma) \le 0; \text{ for sufficiently large } n. \qquad (17)$$

By Lemma 2.1, the delay difference equation

$$\Delta z(n) + \frac{q(n)}{a(n-\sigma)}z(n-\sigma) = 0 \qquad (18)$$

has an eventually positive solution as well. On the other hand, from Lemmas 2.2 and 2.3, we have that (12) or (13) implies that (18) cannot have an eventually positive solution. This contradicts the fact that $z(n) > 0$ and this completes the proof.

Theorem 3.2. Assume that $r(n) \equiv 1$. Assume further that

$$\lim_{n\to\infty} \inf \sum_{s=n-\tau}^{n-1} \frac{p(s-\sigma)q(s)}{a(s-\tau-\sigma)a(s-\sigma)} > \left(\frac{\tau}{\tau+1}\right)^{\tau+1} \qquad (19)$$

Or

$$lim_{n\to\infty} sup \sum_{s=n}^{n+\tau} \frac{p(s-\sigma)q(s)}{a(s-\tau-\sigma)a(s-\sigma)} > 1. \qquad (20)$$

Then every solution of (1) is oscillatory.

Proof. On the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1). Let $\{z(n)\}$ be its associated sequences of $\{x(n)\}$ defined by (10). Then by Lemma 2.4, we have eventually $z(n) > 0$ and $\Delta z(n) < 0$ repeating the procedure as we followed in the proof of Theorem 3.1, we again attain the equation (16). Set

$$\lambda(n) = \frac{-\Delta z(n)}{z(n)} \qquad (21)$$

Then

$$\lambda(n) > 0, \qquad (22)$$

sufficiently large n.

Substituting (21) in (16), we get

$$-\lambda(n)z(n) - \frac{p(n-\sigma)}{a(n-\sigma)}\frac{q(n)}{q(n-\tau)}(-\lambda(n-\tau)z(n-\tau)) + \frac{q(n)}{a(n-\sigma)}z(n-\sigma) = 0$$

or

$$\lambda(n) = \lambda(n-\tau)\frac{p(n-\sigma)}{a(n-\sigma)}\frac{q(n)}{q(n-\tau)}\frac{z(n-\tau)}{z(n)} + \frac{q(n)}{a(n-\sigma)}\frac{z(n-\sigma)}{z(n)} \qquad (23)$$

Now, (23) implies

$$\lambda(n) \ge \frac{q(n)}{a(n-\sigma)}\frac{z(n-\sigma)}{z(n)} \qquad (24)$$

Using the decreasing nature of $\{z(n)\}$ and (24) in (23) we obtain

$$\lambda(n) \ge \frac{p(n-\sigma)q(n)}{a(n-\tau-\sigma)a(n-\sigma)}\frac{z(n-\tau)}{z(n)}$$

or

$$\Delta z(n) + \frac{p(n-\sigma)q(n)}{a(n-\tau-\sigma)a(n-\sigma)}z(n-\tau) \le 0. \qquad (25)$$

Then, by Lemma 2.1, the delay difference equation

$$\Delta z(n) + \frac{p(n-\sigma)q(n)}{a(n-\tau-\sigma)a(n-\sigma)}z(n-\tau) = 0 \qquad (26)$$

has an eventually positive solution as well. On the other hand, by Lemmas 2.2 and 2.3, we have that (19) or (20) implies that (26) cannot have an eventually positive solution. This contradicts the fact that $z(n) > 0$. The proof is complete.

Theorem 3.3. *Assume that $r(n) \equiv 1$ and $\tau \geq \sigma$. Assume that either*

$$\lim_{n\to\infty} \inf \sum_{s=n-\sigma}^{n-1} \left[\frac{p(s-\sigma)q(s)}{a(s-\tau-\sigma)a(s-\sigma)} + \frac{q(s)}{q(s-\sigma)}\right] > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \qquad (27)$$

or

$$\lim_{n\to\infty} \sup \sum_{s=n}^{n+\sigma} \left[\frac{p(s-\sigma)q(s)}{a(s-\tau-\sigma)a(s-\sigma)} + \frac{q(s)}{q(s-\sigma)}\right] > 1. \qquad (28)$$

Then every solution of (1) is oscillatory.
Proof. For the sake of obtaining a contradiction, without loss of generality we assume that there is an eventually positive solution $\{x(n)\}$ of (1). Let$\{z(n)\}$be the associated sequence of $\{x(n)\}$ defined by (10). Then by Lemma 2.4, $\{z(n)\}$ is eventually positive and eventually decreasing. Proceeding as in the proof of Theorem 3.1, we again obtain (23) and (24), using the decreasing nature of $\{z(n)\}$ in (24), we get

$$\lambda(n-\tau) \geq \frac{q(n-\tau)}{a(n-\tau-\sigma)} \qquad (29)$$

Using (29) in (23), we see that $\{z(n)\}$ is a positive solution of the delay difference inequality

$$\Delta z(n) + \frac{p(n-\sigma)q(n)}{a(n-\tau-\sigma)a(n-\sigma)} z(n-\tau) + \frac{q(n)}{a(n-\sigma)} z(n-\sigma) \leq 0. \qquad (30)$$

Since $\Delta z(n) < 0$ and $\tau \geq \sigma$, (30) yields.

$$\Delta z(n) + \left[\frac{p(n-\sigma)q(n)}{a(n-\tau-\sigma)a(n-\sigma)} + \frac{q(n)}{q(n-\sigma)}\right] z(n-\sigma) \leq 0 \qquad (31)$$

Then by Lemma 1, the delay difference equation

$$\Delta z(n) + \left[\frac{p(n-\sigma)q(n)}{a(n-\tau-\sigma)a(n-\sigma)} + \frac{q(n)}{q(n-\sigma)}\right] z(n-\sigma) = 0 \qquad (32)$$

has an eventually positive solution as well. On the other hand, by Lemma 2.2 and 2.3, we have that (28) or (29) implies that (32) cannot have an eventually positive solution.
This contradicts the fact that $z(n) > 0$. The proof is complete.

Theorem 3.4. *Assume that $r(n) \equiv 1$ and $\tau \geq \sigma$. Assume further that either*

$$\lim_{n\to\infty} \inf \sum_{s=n-\sigma}^{n-1} \left[\frac{p(s-\sigma)q(s)}{a(s-\sigma)a(s-\tau-\sigma)} + \frac{q(s)}{q(s-\sigma)}\right] > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}, \qquad (33)$$

or

$$\lim_{n\to\infty} \sup \sum_{s=n-\sigma}^{n+\sigma} \left[\frac{p(s-\sigma)q(s)}{a(s-\sigma)a(s-\tau-\sigma)} + \frac{q(s)}{q(s-\sigma)}\right] > 1. \qquad (34)$$

Then every solution of (1) is oscillatory.
Proof. To obtaining a contradiction, , without loss of generality we may suppose that there is an eventually positive solution $\{x(n)\}$ of (1). Let $\{z(n)\}$ be the associated sequence of $\{x(n)\}$ defined by (10). Then by Lemma 2.4, eventually $z(n) > 0$ and $\{z(n)\}$ is eventually decreasing. Proceeding as in the proof of Theorem 3.2, we obtain (23). From (23) and using the decreasing nature of $\{z(n)\}$,we have (29).
Using (29) in (23) and applying the decreasing nature of $\{z(n)\}$ we have

$$\lambda(n) \geq \frac{q(n)}{a(n-\sigma)} \qquad (35)$$

or

$$\lambda(n-\tau) \geq \frac{q(n-\tau)}{a(n-\sigma-\tau)}. \qquad (36)$$

Using (36) in (23), we have

$$\lambda(n) \geq \frac{p(n-\sigma)q(n)}{a(n-\sigma-\tau)a(n-\sigma)} \frac{z(n-\tau)}{z(n)} + \frac{q(n)}{a(n-\sigma)} \frac{z(n-\tau)}{z(n)} \qquad (37)$$

From (21) and (37) and using the decreasing nature $\{z(n)\}$, we see that $\{z(n)\}$ is a positive solution of the delay difference inequality

$$\Delta z(n) + \left[\frac{p(n-\sigma)q(n)}{a(n-\sigma)a(n-\tau-\sigma)} + \frac{q(n)}{a(n-\sigma)}\right] z(n-\sigma) \leq 0. \qquad (38)$$

Then by Lemma 2.1, the delay difference equation

$$\Delta z(n) + \left[\frac{p(n-\sigma)q(n)}{a(n-\sigma)a(n-\tau-\sigma)} + \frac{q(n)}{a(n-\sigma)}\right] z(n-\sigma) = 0. \qquad (39)$$

has an eventually positive solution as well. On the other hand, by Lemma 2.2 and 2.3, we have that (33) or (34) implies that (39) cannot have an eventually positive solution.
This contradiction the fact that $z(n) > 0.$ The proof is complete.

Theorem 3.5. *Assume that $\tau \geq \sigma$. Assume further that either*

$$\lim_{n\to\infty} \inf \sum_{s=n-\sigma}^{n-1} \frac{q(s)}{\tau(s-\sigma)a(s-\sigma)} > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \quad (40)$$

or

$$\lim_{n\to\infty} \sup \sum_{s=n}^{n+\sigma} \frac{q(s)}{\tau(s-\sigma)a(s-\sigma)} > 1. \quad (41)$$

Then every solution of (1) is oscillatory.
Proof. To obtaining a contradiction, , without loss of generality we may assume that there is an eventually positive solution $\{x(n)\}$ of (1). Let $\{z(n)\}$ be the associated sequence of $\{x(n)\}$ defined by (10). Then by Lemma 2.4, eventually $z(n) > 0$ and eventually $\{z(n)\}$ is decreasing. From (10), we have
$z(n) \leq a(n)r(n)$
or

$$x(n) \geq \frac{z(n)}{a(n)} \qquad (42)$$

From (42) and (1), we have

$$\Delta\big(r(n)z(n)\big) + \frac{q(n)}{a(n-\sigma)} z(n-\sigma) \leq 0. \qquad (43)$$

Set

$$y(n) = r(n)z(n). \qquad (44)$$

Then, from (42) we get

$$\Delta y(n) + \frac{q(n)}{r(n-\sigma)a(n-\sigma)} y(n-\sigma) \leq 0. \qquad (45)$$

Then, by Lemma 2.1, the delay difference equation

$$\Delta y(n) + \frac{q(n)}{r(n-\sigma)a(n-\sigma)} y(n-\sigma) = 0. \qquad (46)$$

has an eventually positive as well. On the other hand, by Lemmas 2.2 and 2.3, we have that (42) or (43) implies that (46) cannot have an eventually positive solution. This contradicts the fact that $y(n) > 0.$ The proof is complete.

IV. SOME EXAMPLES

Example 4.1. *Consider the following first order nonlinear neutral delay difference equation*

$$\Delta\left[x(n) - \frac{1}{n+1}x(n-1)\right] + \frac{1}{2}\left(2 + \frac{1}{n+1} + \frac{1}{n+2}\right)x(n-2) = 0; \quad n = 2,3,\ldots \qquad (47)$$

we can see that, $\tau = 1, \sigma = 2, a(n) = 1, r(n) = 1, p(n) = \frac{1}{n+1},$
$q(n) = \frac{1}{2}\left(2 + \frac{1}{n+1} + \frac{1}{n+2}\right).$

Also,

$$\liminf_{n\to\infty} \sum_{s=n-2}^{n-1} \frac{q(s)}{a(s-\sigma)} = \liminf_{n\to\infty} \frac{1}{2}\left[4 + \frac{2}{n} + \frac{1}{n-1} + \frac{1}{n+1}\right]$$
$$= 2$$
$$> \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} = \left(\frac{2}{3}\right)^3.$$

Hence all conditions of the Theorem 3.1 are satisfied. Therefore every solution of (47) is oscillatory. One of its such solution is $x(n) = (-1)^n$.

Example 4.2. *Consider the following the first order neutral delay difference equation*

$$\Delta\left[\frac{(n+3)}{n+2}x(n)-\frac{1}{n+2}x(n-1)\right]+\left(\frac{n^2+n+2}{n-1}\right)x(n-2)=0;\ \ n=3,4,\ldots \quad (48)$$

Clearly $\tau=1,\ \ \sigma=2,\ \ a(n)=\frac{n+3}{n+2},\ \ p(n)=\frac{1}{n+2},\ \ q(n)=\frac{n^2+n+2}{n-1}$.

and , $\frac{p(n)}{a(n)}=\frac{1}{n+3}\leq\frac{1}{6}<1.$

Also

$$\limsup_{n\to\infty}\sum_{s=n}^{n+2}\frac{q(s)p(s-\sigma)}{a(s-\tau-\sigma)a(s-\sigma)}$$

$$=\limsup_{n\to\infty}\left[3+\frac{2}{n+n^2}+\frac{2}{(n+1)+(n+1)^2}+\frac{1}{(n+2)+(n+2)^2}\right]$$

$= 3 > 1.$

Thus all the conditions of the Theorem 4.2 are satisfied and hence every solution of (48) is oscillatory.

Example 4.3. Consider the following first order neutral delay difference equation

$$\Delta\left[n\left(\frac{n+1}{n}x(n)-\frac{1}{n}x(n-1)\right)\right]+nx(n-2)=0;\ \ n=3,4,5,\ldots \quad (49)$$

Here, $\tau=1,\ \ \sigma=2,\ \ r(n)=n,\ \ a(n)=\frac{n+1}{n},\ \ p(n)=\frac{1}{n},\ \ q(n)=n$.

We can easily verify that $\Delta r(n)=1>1, 0<a(n)\leq\frac{4}{3},\ \ \frac{p(n)}{a(n)}=\frac{1}{n+1}\leq\frac{1}{4}<1.$

Also

$$\liminf_{n\to\infty}\sum_{s=n-\sigma}^{n-1}\frac{q(s)}{r(s-\sigma)a(s-\sigma)}$$

$$=\liminf_{n\to\infty}\sum_{s=n-2}^{n-1}\left[1+\frac{1}{s-1}\right]$$

$$=\liminf_{n\to\infty}\sum_{s=n-2}^{n-1}\left[2+\frac{1}{n-3}+\frac{1}{n-2}\right]$$

$$=2>\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}=\left(\frac{2}{3}\right)^3.$$

Hence all the conditions of the Theorem 3.5 are satisfied and hence all the solution of (49) oscillates.

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