International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 3 Issue 7 || November. 2015 || PP-21-34

Laplace Transforms of Some Special Functions of Mathematical Physics Using Mellin-Barnes Type Contour Integration

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Abstract: *In this paper, we find some results on Laplace transforms of Special functions like products of Ordinary Bessel's function of I kind and Modified Bessel's function of I kind, five Complete Elliptic integrals, Hyper-Bessel function of Humbert, Modified Hyper-Bessel function of Delerue, Error function, Incomplete Beta function, Incomplete Gamma function, Fresnel Integrals, Sine Integral, Hyperbolic Sine Integral, Polylogarithm function, Struve function, Modified Struve function, Lommel function, Kelvin's functions and Lerch's Transcendent, in terms of Meijer's G-function of one variable and generalized hypergeometric function of one variable, using Mellin-Barnes type contour integral technique. The results presented here are presumably new.*

Keywords and Phrases: *Generalized Hypergeometric function, Laplace transforms, Meijer's G - function, Gauss's multiplication formula, Legendre's duplication formula and triplication formula.*
2010 MSC(AMS) *Primary:44A10; Secondary: 33C20, 33C60.*

I. Introduction and Preliminaries

Throughout our present paper, we use the following standard notations:

$\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \setminus \{0\}$.

Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & ;(\nu = 0; \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & ;(\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases} \tag{1.1}$$

it is being understood *conventionally* that $(0)_0 := 1$ and assumed tacitly that the Gamma quotient exists.

Of all the integrals which contain gamma functions in their integrands the most important ones are the so-called Mellin-Barnes integrals. Such integrals were first introduced by S. Pincherle in the year 1888; their theory has been developed by H. Mellin in the year 1910 and they were used for a complete integration of the hypergeometric differential equation by E.W. Barnes in the year 1908. The generalized Hypergeometric function [37, p. 100 Th. 35 and p. 102 Th. 36] is defined by means of Mellin-Barnes type contour integral in the following form, when

$p \leq q+1$ then

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}{\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}\,{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array} z\right]=\frac{1}{2\pi\omega}\int_{L_1}\frac{(-z)^{\xi}\Gamma(-\xi)\Gamma(\alpha_1+\xi),\ldots,\Gamma(\alpha_p+\xi)}{\Gamma(\beta_1+\xi),\ldots,\Gamma(\beta_q+\xi)}d\xi \tag{1.2}$$

where L_1 is a suitable Mellin-Barnes path of integration [See 37, p. 95 figure (5), p. 98 figure(6)]and $\alpha_i,\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\{i=1,2,3,\ldots,p,\ j=1,2,3,\ldots,q\}$ and $\omega=\sqrt{-1}$, $z\neq 0$.

When $p=q+1$ then $|arg(-z)|<\pi$ and suppose that $|z|<1$.

When $p=q$ then $|arg(-z)|<\frac{\pi}{2}$ i.e. $\Re(z)<0$.

When $p<q$ then equation (1.2) is also valid.

Gauss's multiplication formula for the product of gamma functions [37, p.26 Th. 10; 43, p.23 (27)], is given by .

$$\Gamma(mz)=(2\pi)^{\frac{1-m}{2}}m^{(mz-\frac{1}{2})}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)\ldots\Gamma\left(z+\frac{m-1}{m}\right) \tag{1.3}$$

where m is positive integer; $mz\in\mathbb{C}\backslash\mathbb{Z}_0^-$. For $m=2$ and $m=3$ we get Legendre's duplication formula and triplication formula respectively.

Using formula (1.3), we obtain

$$\Gamma(c+ks)=\Gamma\left\{k\left(s+\frac{c}{k}\right)\right\}=(2\pi)^{\frac{1-k}{2}}k^{(c-\frac{1}{2}+ks)}\prod_{q=1}^{k}\Gamma\left(s+\frac{c+q-1}{k}\right) \tag{1.4}$$

where $k=1,2,3,\ldots$ and $c+ks\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

In an attempt to give a meaning to the symbol ${}_pF_q$, when $p>q+1$, Meijer introduced the G-function into Mathematical Analysis. Firstly the G-function was defined by Meijer [24] in the year 1936 by means of a finite series of generalized hypergeometric functions. Later on the Meijer's G-function of order (m,n,p,q) was defined by means of Mellin-Barnes type contour integral formula [11, p.207 (5.3.1); see also 19, p.143 (5.2.1); 23, p.2 (1.1.1, 1.1.3); 25, p.83; 26, p.1064 (21)], in the following form

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{array}{c}a_1,a_2,\ldots,a_n;a_{n+1},\ldots,a_p\\ b_1,b_2,\ldots,b_m;b_{m+1},\ldots,b_q\end{array}\right)=\frac{1}{2\pi\omega}\int_{L_2}\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s ds \tag{1.5}$$

$$(a_k-b_j\neq 1,2,3,\ldots;\ k=1,2,3,\ldots,n \text{ and } j=1,2,3,\ldots,m)$$

where $0\leq m\leq q, 0\leq n\leq p\,; z\neq 0$ and L_2 is a suitable contour (See three cases of contour in the monographs [11, p.207 (2,3,4); 19, p.144 (2,3,4); 32, p.617 (1,2,3,4)]) and an empty product is interpreted as 1 and the parameters are such that no pole of $\Gamma(b_j-s)$, $j=1,2,3,\ldots,m$ coincides with any pole of $\Gamma(1-a_k+s)$, $k=1,2,3,\ldots,n$. Without any loss of generality, we are assuming that $p\leq q$. The MacRobert's E-function is the particular case of Meijer's

G-function.

The G-function is symmetric with respect to order of the parameters in four groups

$a_1, a_2, \ldots, a_n$; $a_{n+1}, \ldots, a_p$; $b_1, b_2, \ldots, b_m$; $b_{m+1}, \ldots, b_q$ individually.

If no pair among the parameters $b_1, b_2, \ldots, b_m$ may differ by an integer or zero (i.e. all poles are of the first order) then Meijer's function $G^{m,n}_{p,q}(z)$ can be expressed as a sum of m-generalized hypergeometric functions ${}_pF_{q-1}\left((-1)^{p-m-n}z\right)$ under the condition ($p < q$ and for all finite values of z) or ($p = q$ and $|z| < 1$).

If no pair among the parameters $a_1, a_2, \ldots, a_n$ may differ by an integer or zero (i.e. all poles are of the first order) then Meijer's function $G^{m,n}_{p,q}(z)$ can be expressed as a sum of n-generalized hypergeometric functions ${}_qF_{p-1}\left((-1)^{q-m-n}z^{-1}\right)$ under the condition ($p > q$ and for all finite values of z) or ($p = q$ and $|z| > 1$).

If one (or more) pair among the parameters $a_1, a_2, \ldots, a_n$ or $b_1, b_2, \ldots, b_m$ may differ by an integer or zero then Logarithmic forms of the G-function occur due to the appearance of the poles of the higher order than unity, in the integrand of contour integral (1.5).

Property for cancellation of the numerator and denominator parameters

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_{q-1}, a_1 \end{matrix}\right) = G^{m,n-1}_{p-1,q-1}\left(z \,\middle|\, \begin{matrix} a_2, a_3, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_{q-1} \end{matrix}\right) \tag{1.6}$$

where $n, p, q \geq 1$.

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{p-1}, b_1 \\ b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix}\right) = G^{m-1,n}_{p-1,q-1}\left(z \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{p-1} \\ b_2, b_3, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix}\right) \tag{1.7}$$

where $m, p, q \geq 1$.

Translation property [26, p.1066 (24)]

$$z^{\sigma} G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix}\right) = G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1+\sigma, a_2+\sigma, \ldots, a_n+\sigma, a_{n+1}+\sigma, \ldots, a_p+\sigma \\ b_1+\sigma, b_2+\sigma, \ldots, b_m+\sigma, b_{m+1}+\sigma, \ldots, b_q+\sigma \end{matrix}\right)$$

Symmetric property (Transformation formula)

$$G_{p,q}^{m,n}\left(z\left|\begin{array}{l}a_1,a_2,\ldots,a_n,a_{n+1},\ldots,a_p\\ b_1,b_2,\ldots,b_m,b_{m+1},\ldots,b_q\end{array}\right.\right)=G_{q,p}^{n,m}\left(\frac{1}{z}\left|\begin{array}{l}1-b_1,1-b_2,\ldots,1-b_m,1-b_{m+1},\ldots,1-b_q\\ 1-a_1,1-a_2,\ldots,1-a_n,1-a_{n+1},\ldots,1-a_p\end{array}\right.\right) \tag{1.8}$$

Reduction formula between ${}_pF_q$ and G-functions [11, p.215 (5.6.1); 43, p.47 (9)], is given by when $p \leq q+1$, then

$$\frac{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_p)}{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_q)}{}_pF_q\left[\begin{array}{l}a_1,a_2,\ldots,a_p;\\ b_1,b_2,\ldots,b_q;\end{array} z\right]=G_{p,q+1}^{1,p}\left(-z\left|\begin{array}{l}1-a_1,1-a_2,\ldots,1-a_p\\ 0,1-b_1,1-b_2,\ldots,1-b_q\end{array}\right.\right)$$

$$=G_{q+1,p}^{p,1}\left(-\frac{1}{z}\left|\begin{array}{l}1,b_1,b_2,\ldots,b_q\\ a_1,a_2,\ldots,a_p\end{array}\right.\right)=(-z)^1G_{p,q+1}^{1,p}\left(-z\left|\begin{array}{l}-a_1,-a_2,\ldots,-a_p\\ -1,-b_1,-b_2,\ldots,-b_q\end{array}\right.\right) \tag{1.9}$$

where ($p \leq q$ and $|z| < \infty$) or ($p = q+1$ and $|z| < 1$).

For hypergeometric forms of Special functions of Mathematical Physics, we refer the monographs [1-7,11,12,14-16,18-22,27,30,32,37,39,40,42,43,45,46].

II. Laplace transform

If there exists a number "M" independent of t so that $|\frac{f(t)}{g(t)}| \leq M$ as $t \longrightarrow t_0$ in the region R of the complex z-plane, where $g(t) \neq 0$, then we say that $f(t) = O[g(t)]$ as $t \longrightarrow t_0$ in R.

If $\lim_{t\longrightarrow t_0}\frac{f(t)}{g(t)} = 0$, then we say that $f(t) = o[g(t)]$ as $t \longrightarrow t_0$ in R.

Let the function $f(t)$ be piecewise continuous on the closed interval $0 \leq t \leq T$ for every finite $T > 0$. Also let

$$f(t) = O[e^{\alpha t}], \qquad t \to \infty \tag{2.1}$$

for some α. Operational images (or operational representations) of many classes of special functions in the classical Laplace transform

$$\mathfrak{L}\{f(t) : p\} = \int_0^\infty e^{-pt}f(t)dt = F(p), \qquad \Re(p) > \alpha, \tag{2.2}$$

can be obtained by appealing to Euler's integral

$$\int_0^\infty e^{-pt}t^{\lambda-1}dt = \frac{\Gamma(\lambda)}{p^\lambda} \tag{2.3}$$

where $\min\{\Re(\lambda), \Re(p)\} > 0$ or $(\Re(p) = 0, 0 < \Re(\lambda) < 1)$.

The integral (2.2) appeared for the first time in Euler's investigation in the year 1737. The regular use of the transformation of the form (2.2) began after the publication of P. S. Laplace's book in the year 1812. At the present time the Laplace transformation (2.2) is the most usable integral transformation. A complete account or elements of the theory of Laplace transformation can be found in numerous books on Laplace transformation, on operational calculus or on integral transformations. Among them we mention the monographs [8,10,13,28,29,33-36,38,41,44,47].

III. Theorem on Laplace Transforms

Statement:

When $A \leq B+1$, $\Re(p) > 0$, $\Re(c) > 0$ and k is a positive integer. Suppose (a_A) abbreviates the array of A parameters given by $a_1, a_2, \ldots, a_A$ with similar interpretation for (b_B), then

$$\mathfrak{L}\left\{ t^{c-1}{}_AF_B\left[\begin{array}{c} (a_A); \\ (b_B); \end{array} yt^k\right] : p\right\}$$

$$= \frac{(2\pi)^{\frac{1-k}{2}}}{p^c}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)} k^{(c-\frac{1}{2})} G^{A+k,1}_{1+B,A+k}\left(\frac{-p^k}{yk^k}\middle| \begin{array}{c} 1, b_1, b_2, \ldots, b_B \\ a_1, a_2, \ldots, a_A, \frac{c}{k}, \frac{c+1}{k}, \ldots, \frac{c+k-1}{k} \end{array}\right) \tag{3.1}$$

$$= \frac{(2\pi)^{\frac{1-k}{2}}}{p^c}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)} k^{(c-\frac{1}{2})} \times$$

$$\times\, G^{1,A+k}_{A+k,1+B}\left(\frac{-yk^k}{p^k}\middle| \begin{array}{c} 1-a_1, 1-a_2, \ldots, 1-a_A, 1+\left(\frac{-c}{k}\right), 1+\left(\frac{-c-1}{k}\right), \ldots, 1+\left(\frac{-c-k+1}{k}\right) \\ 0, 1-b_1, 1-b_2, \ldots, 1-b_B \end{array}\right) \tag{3.2}$$

provided that the right hand sides of (3.1) and (3.2) are convergent [See 11, p.207 (2,3,4); 19, p.144 (2,3,4); 32, p.617 (1,2,3,4)] and $a_1, a_2, \ldots, a_A, b_1, b_2, \ldots, b_B \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Proof: Consider the left hand side of equation (3.1):

$$\mathfrak{L}\left\{ t^{c-1}{}_AF_B\left[\begin{array}{c} (a_A); \\ (b_B); \end{array} yt^k\right] : p\right\} = \int_0^\infty e^{-pt}t^{c-1}{}_AF_B\left[\begin{array}{c} (a_A); \\ (b_B); \end{array} yt^k\right] dt$$

$$= \int_0^\infty e^{-pt}t^{c-1}\left(\frac{1}{2\pi\omega}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}\int_L \frac{(-yt^k)^s\Gamma(-s)\prod_{i=1}^{A}\Gamma(a_i+s)}{\prod_{j=1}^{B}\Gamma(b_j+s)} ds\right) dt$$

where L is a suitable Mellin-Barnes type contour.

Now changing the order of integration, we get

$$\mathfrak{L}\left\{ t^{c-1}{}_AF_B\left[\begin{array}{c} (a_A); \\ (b_B); \end{array} yt^k\right] : p\right\} = \frac{1}{2\pi\omega}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}\int_L\left(\frac{(-y)^s\Gamma(-s)\prod_{i=1}^{A}\Gamma(a_i+s)}{\prod_{j=1}^{B}\Gamma(b_j+s)}\left(\int_0^\infty e^{-pt}t^{c+ks-1}dt\right)\right) ds$$

$$= \frac{1}{p^c}\frac{1}{2\pi\omega}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}\int_L \frac{\left(\frac{-y}{p^k}\right)^s\Gamma(-s)\Gamma(c+ks)\prod_{i=1}^{A}\Gamma(a_i+s)}{\prod_{j=1}^{B}\Gamma(b_j+s)} ds$$

Now using Gauss's multiplication formula (1.4) for $\Gamma(c+ks)$, we get

$$\mathfrak{L}\left\{t^{c-1}{}_AF_B\left[\begin{array}{cc}(a_A); & \\ & yt^k\\ (b_B); & \end{array}\right]:p\right\}=\frac{(2\pi)^{\frac{1-k}{2}}}{p^c}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}k^{(c-\frac{1}{2})}\times$$

$$\times\frac{1}{2\pi\omega}\int_L\frac{\Gamma(-s)\prod_{i=1}^{A}\Gamma(a_i+s)\prod_{q=1}^{k}\Gamma\left(s+\frac{c+q-1}{k}\right)}{\prod_{j=1}^{B}\Gamma(b_j+s)}\left(\frac{-yk^k}{p^k}\right)^s ds$$

Using definition (1.5) and transformation formula (1.8) of G-function, we get main results (3.2) and (3.1) respectively.

IV. Laplace Transforms of Special Functions of Mathematical Physics

For convenience, we shall use the notation $\Delta(N;\lambda)$ for array of N parameters given by $\frac{\lambda}{N},\frac{\lambda+1}{N},\frac{\lambda+2}{N},\ldots,\frac{\lambda+N-1}{N}$. The following formulas hold for those suitable values of parameters for which gamma factors of the numerator and denominator are finite. The Laplace transforms of following Special functions are not found in the available literature on Laplace transforms [8,10,13,28,29,33-36,38,41,44,47]. Making suitable adjustment of parameters and variables in equation (3.1), using the cancellation property (1.7), applying reduction formula (1.9), Legendre's duplication formula and triplication formula for the product of Gamma functions, after simplification we can find the following results, valid under the conditions associated with the result (3.1).

Case(1): Put $c=\mu+\nu+1, A=2, B=3, a_1=\dfrac{\mu+\nu+1}{2}, a_2=\dfrac{\mu+\nu+2}{2}, b_1=\mu+1, b_2=\nu+1, b_3=\mu+\nu+1, y=-1, k=2$, in equation (3.1), we have

$$\mathfrak{L}\left\{J_\mu(t)J_\nu(t):p\right\}=\frac{2^{\mu+\nu}}{\pi p^{\mu+\nu+1}}G^{4,1}_{4,4}\left(\frac{p^2}{4}\left|\begin{array}{c}1,\mu+1,\nu+1,\mu+\nu+1\\ \frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2},\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2}\end{array}\right.\right)$$

$$=\frac{\Gamma(\mu+\nu+1)}{2^{\mu+\nu}\,p^{\mu+\nu+1}\Gamma(\mu+1)\Gamma(\nu+1)}{}_4F_3\left[\begin{array}{cc}\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2},\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2}; & \\ & -\frac{4}{p^2}\\ \mu+1,\quad\nu+1,\quad\mu+\nu+1; & \end{array}\right]\tag{4.1}$$

where the product $J_\mu(t)J_\nu(t)$ is given in the monograph [19,p.216 (39)] and $\mu+\nu+1,\mu+1,\nu+1\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(2): Put $c = 2\nu + 2, A = 1, B = 2, a_1 = \nu + \dfrac{3}{2}, b_1 = \nu + 2, b_2 = 2\nu + 2, y = -1, k = 2$, in equation (3.1), we have

$$\mathfrak{L}\left\{J_\nu(t)J_{\nu+1}(t) : p\right\} = \frac{2^{2\nu+1}}{\pi p^{2\nu+2}} G^{3,1}_{3,3}\left(\frac{p^2}{4}\middle|\begin{matrix} 1, \nu+2, 2\nu+2 \\ \nu+\frac{3}{2}, \nu+1, \nu+\frac{3}{2}\end{matrix}\right)$$

$$= \frac{\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}\ p^{2\nu+2}\ \Gamma(\nu+2)} {}_3F_2\left[\begin{matrix} \nu+1, \nu+\frac{3}{2}, \nu+\frac{3}{2}; \\ \nu+2, \quad 2\nu+2 \quad ; \end{matrix} -\frac{4}{p^2}\right] \tag{4.2}$$

where the product $J_\nu(t)J_{\nu+1}(t)$ is given in the monograph [19,p.216 (40)] and $\nu + \frac{3}{2}, \nu + 2 \ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(3): Put $c = 2\nu + 1, A = 1, B = 2, a_1 = \nu + \dfrac{1}{2}, b_1 = \nu + 1, b_2 = 2\nu + 1, y = -1, k = 2$, in equation (3.1), we have

$$\mathfrak{L}\left\{J_\nu^2(t) : p\right\} = \frac{2^{2\nu}}{\pi p^{2\nu+1}} G^{2,1}_{2,2}\left(\frac{p^2}{4}\middle|\begin{matrix} 1, 2\nu+1 \\ \nu+\frac{1}{2}, \nu+\frac{1}{2}\end{matrix}\right)$$

$$= \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\ p^{2\nu+1}\ \Gamma(\nu+1)} {}_2F_1\left[\begin{matrix} \nu+\frac{1}{2}, \nu+\frac{1}{2}; \\ 2\nu+1 \quad ; \end{matrix} -\frac{4}{p^2}\right] \tag{4.3}$$

where the product $J_\nu^2(t)$ is given in the monograph [19,p.216 (41)] and $\nu + \frac{1}{2}, \nu + 1 \ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Similarly we can obtain

$$\mathfrak{L}\left\{J_\nu(t)J_{-\nu}(t) : p\right\} = \frac{1}{\pi\ p} G^{3,1}_{3,3}\left(\frac{p^2}{4}\middle|\begin{matrix} 1, -\nu+1, \nu+1 \\ \frac{1}{2}, \quad \frac{1}{2}, \quad 1\end{matrix}\right)$$

$$= \frac{\sin(\nu\ \pi)}{\pi\ p\ \nu} {}_3F_2\left[\begin{matrix} \frac{1}{2}, \quad \frac{1}{2}, \quad 1 \quad ; \\ -\nu+1, \nu+1; \end{matrix} -\frac{4}{p^2}\right] \tag{4.4}$$

where $\nu \neq 0, \pm 1, \pm 2, \ldots$.

Case(4): Put $c = 2\nu + 1, A = 0, B = 3, b_1 = \dfrac{\nu+1}{2}, b_2 = \dfrac{\nu+2}{2}, b_3 = \nu + 1, y = -\dfrac{1}{64}, k = 4$, in equation (3.1), we have

$$\mathfrak{L}\left\{J_\nu(t)I_\nu(t) : p\right\} = \frac{2^{\nu-\frac{1}{2}}}{\pi p^{2\nu+1}} G^{2,1}_{2,2}\left(\frac{p^4}{4}\middle|\begin{matrix} 1, \nu+1 \\ \frac{2\nu+1}{4}, \frac{2\nu+3}{4}\end{matrix}\right)$$

$$= \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\; p^{2\nu+1}\;\Gamma(\nu+1)}\, {}_2F_1\left[\begin{array}{c} \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \\ \nu+1 \quad ; \end{array} -\frac{4}{p^4}\right] \tag{4.5}$$

where the product $J_\nu(t)I_\nu(t)$ is given in the monograph [19,p.216 (43)] and $\nu+\frac{1}{2}, \nu+1 \ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(5): Put $c = m+n+1, A = 0, B = 2, b_1 = m+1, b_2 = n+1, y = -\dfrac{1}{27}, k = 3$, in equation (3.1), we have

$$\mathfrak{L}\{J_{m,n}(t) : p\} = \frac{\sqrt{3}}{p^{m+n+1}\; 2\pi} G^{3,1}_{3,3}\left(p^3 \left| \begin{array}{c} 1, m+1, n+1 \\ \frac{m+n+1}{3}, \frac{m+n+2}{3}, \frac{m+n+3}{3} \end{array}\right.\right)$$

$$= \frac{\Gamma(m+n+1)}{p^{m+n+1}\; 3^{(m+n)}\; \Gamma(m+1)\Gamma(n+1)}\, {}_3F_2\left[\begin{array}{c} \Delta(3; m+n+1); \\ m+1, \quad n+1\; ; \end{array} -\frac{1}{p^3}\right] \tag{4.6}$$

where $J_{m,n}(t)$ is Hyper-Bessel function of Humbert [14, p.250 (19.7.7); 17; 31, p.102] and $m+n+1, m+1, n+1 \ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(6): Put $c = m+n+1, A = 0, B = 2, b_1 = m+1, b_2 = n+1, y = \dfrac{1}{27}, k = 3$, in equation (3.1), we have

$$\mathfrak{L}\{I_{m,n}(t) : p\} = \frac{\sqrt{3}}{p^{m+n+1}\; 2\pi} G^{3,1}_{3,3}\left(-p^3 \left| \begin{array}{c} 1, m+1, n+1 \\ \frac{m+n+1}{3}, \frac{m+n+2}{3}, \frac{m+n+3}{3} \end{array}\right.\right)$$

$$= \frac{\Gamma(m+n+1)}{p^{m+n+1}\; 3^{(m+n)}\; \Gamma(m+1)\Gamma(n+1)}\, {}_3F_2\left[\begin{array}{c} \Delta(3; m+n+1); \\ m+1, \quad n+1\; ; \end{array} \frac{1}{p^3}\right] \tag{4.7}$$

where $I_{m,n}(t)$ is Modified Hyper-Bessel function of Delerue [9] and $m+n+1, m+1, n+1 \ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(7): Put $c = 2, A = 1, B = 1, a_1 = \dfrac{1}{2}, b_1 = \dfrac{3}{2}, y = -1, k = 2$, in equation (3.1), we have

$$\mathfrak{L}\{\text{erf}(t) : p\} = \frac{2}{\pi\; p^2} G^{2,1}_{1,2}\left(\frac{p^2}{4} \left| \begin{array}{c} 1 \\ \frac{1}{2}, 1 \end{array}\right.\right) \tag{4.8}$$

where $\text{erf}(t)$ is Error function [37, p.36 (6), p.127 (Q.1)].

Case(8): Put $c = \alpha + 1, A = 2, B = 1, a_1 = \alpha, a_2 = 1 - \beta, b_1 = 1 + \alpha, y = 1, k = 1$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{B}_t(\alpha, \beta) : p\} = \frac{1}{p^{\alpha+1}\Gamma(1-\beta)} G^{2,1}_{1,2}\left(-p \,\middle|\, \begin{matrix} 1 \\ \alpha, 1-\beta \end{matrix}\right) \tag{4.9}$$

where $\mathbf{B}_t(\alpha, \beta)$ is Incomplete Beta function[43, p.35 (31)] and $1 - \beta \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(9): Put $c = a + 1, A = 1, B = 1, a_1 = a, b_1 = 1 + a, y = -1, k = 1$, in equation (3.1), we have

$$\mathfrak{L}\{\gamma(a, t) : p\} = \frac{1}{p^{a+1}} G^{1,1}_{1,1}\left(p \,\middle|\, \begin{matrix} 1 \\ a \end{matrix}\right)$$

$$= \frac{\Gamma(a)}{p^{a+1}} {}_1F_0\left[\begin{matrix} a\,; \\ -\,; \end{matrix} \; -\frac{1}{p}\right] = \frac{\Gamma(a)}{p}(1+p)^{-a} \tag{4.10}$$

where $\gamma(a, t)$ is Incomplete Gamma function[19, p.220(6.2.11.1); 37, p.127 (Q.2)] and $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(10): Put $c = 1, A = 2, B = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{2}, b_1 = 2, y = 1, k = 2$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{B}(t) : p\} = \frac{1}{4p\sqrt{\pi}} G^{4,1}_{2,4}\left(-\frac{p^2}{4} \,\middle|\, \begin{matrix} 1, 2 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \end{matrix}\right) \tag{4.11}$$

where $\mathbf{B}(t)$ is complete Elliptic integral [12, p.321 13.8(25)].

Case(11): Put $c = 1, A = 2, B = 1, a_1 = \frac{3}{2}, a_2 = \frac{3}{2}, b_1 = 3, y = 1, k = 2$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{C}(t) : p\} = \frac{1}{2p\sqrt{\pi}} G^{4,1}_{2,4}\left(-\frac{p^2}{4} \,\middle|\, \begin{matrix} 1, 3 \\ \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, 1 \end{matrix}\right) \tag{4.12}$$

where $\mathbf{C}(t)$ is complete Elliptic integral [12, p.321 13.8(25)].

Case(12): Put $c = 1, A = 2, B = 1, a_1 = \frac{1}{2}, a_2 = \frac{3}{2}, b_1 = 2, y = 1, k = 2$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{D}(t) : p\} = \frac{1}{2p\sqrt{\pi}} G^{4,1}_{2,4}\left(-\frac{p^2}{4} \,\middle|\, \begin{matrix} 1, 2 \\ \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, 1 \end{matrix}\right) \tag{4.13}$$

where $\mathbf{D}(t)$ is complete Elliptic integral [12, p.321 13.8(25)].

Case(13): Put $c = 1, A = 2, B = 1, a_1 = \frac{1}{2}, a_2 = -\frac{1}{2}, b_1 = 1, y = 1, k = 2$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{E}(t) : p\} = -\frac{1}{4p\sqrt{\pi}} G^{3,1}_{1,3}\left(-\frac{p^2}{4} \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \end{matrix}\right) \tag{4.14}$$

where $\mathbf{E}(t)$ is complete Elliptic integral of second kind [12, pp.317-318 13.8(2)(6)].

Case(14): Put $c = 1, A = 2, B = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{2}, b_1 = 1, y = 1, k = 2$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{K}(t):p\} = \frac{1}{2p\sqrt{\pi}} G^{3,1}_{1,3}\left(-\frac{p^2}{4}\,\middle|\,\begin{matrix} 1 \\ \frac{1}{2},\frac{1}{2},\frac{1}{2}\end{matrix}\right) \tag{4.15}$$

where $\mathbf{K}(t)$ is complete Elliptic integral of first kind [12, pp.317-318 13.8(1)(5)].

Case(16): Put $c = a, A = 2, B = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{2}, b_1 = 1, y = b, k = 1$, in equation (3.1), we have

$$\mathfrak{L}\{t^{a-1}\mathbf{K}(\sqrt{bt}):p\} = \frac{1}{2\,p^a} G^{3,1}_{2,3}\left(-\frac{p}{b}\,\middle|\,\begin{matrix} 1,1 \\ \frac{1}{2},\frac{1}{2},a\end{matrix}\right) \tag{4.16}$$

Case(17): Put $c = 4, A = 1, B = 2, a_1 = \frac{3}{4}, b_1 = \frac{3}{2}, b_2 = \frac{7}{4}, y = -\frac{\pi^2}{16}, k = 4$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{S}(t):p\} = \frac{2\sqrt{2}}{p^4} G^{3,1}_{1,3}\left(\frac{p^4}{16\,\pi^2}\,\middle|\,\begin{matrix} 1 \\ \frac{3}{4},1,\frac{5}{4}\end{matrix}\right) \tag{4.17}$$

where $\mathbf{S}(t)$ is Fresnel integral [1, p.300 (7.3.2)].

Case(18): Put $c = 4, A = 1, B = 2, a_1 = \frac{3}{4}, b_1 = \frac{3}{2}, b_2 = \frac{7}{4}, y = -\frac{1}{4}, k = 4$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{S}_1(t):p\} = \frac{8}{\pi^{\frac{3}{2}}p^4} G^{3,1}_{1,3}\left(\frac{p^4}{64}\,\middle|\,\begin{matrix} 1 \\ \frac{3}{4},1,\frac{5}{4}\end{matrix}\right) \tag{4.18}$$

where $\mathbf{S}_1(t)$ is Fresnel integral [1, p.300 (7.3.4)].

Case(19): Put $c = \frac{5}{2}, A = 1, B = 2, a_1 = \frac{3}{4}, b_1 = \frac{3}{2}, b_2 = \frac{7}{4}, y = -\frac{1}{4}, k = 2$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{S}_2(t):p\} = \frac{1}{2\,\sqrt{\pi}\,p^{\frac{5}{2}}} G^{2,1}_{2,2}\left(p^2\,\middle|\,\begin{matrix} 1,\frac{3}{2} \\ \frac{3}{4},\frac{5}{4}\end{matrix}\right)$$

$$= \frac{1}{2\,\sqrt{2}\,p^{\frac{5}{2}}}\ {}_2F_1\left[\begin{matrix}\frac{3}{4},\frac{5}{4}; \\ \frac{3}{2}\quad ;\end{matrix}\ -\frac{1}{p^2}\right] \tag{4.19}$$

where $\mathbf{S}_2(t)$ is Fresnel integral [1, p.300 (7.3.4)].

Case(19): Put $c = 2, A = 1, B = 2, a_1 = \frac{1}{4}, b_1 = \frac{1}{2}, b_2 = \frac{5}{4}, y = -\frac{\pi^2}{16}, k = 4$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbb{C}^*(t):p\} = \frac{1}{\pi\,\sqrt{2}\,p^2} G^{3,1}_{1,3}\left(\frac{p^4}{16\,\pi^2}\,\middle|\,\begin{matrix} 1 \\ \frac{1}{4},\frac{3}{4},1\end{matrix}\right) \tag{4.20}$$

where $\mathbf{C}^*(t)$ is Fresnel integral [1, p.300 (7.3.1)].

Case(20): Put $c = 2, A = 1, B = 2, a_1 = \frac{1}{4}, b_1 = \frac{1}{2}, b_2 = \frac{5}{4}, y = -\frac{1}{4}, k = 4$, in equation (3.1), we have

$$\mathfrak{L}\left\{\mathbf{C}_1(t):p\right\}=\frac{1}{\pi^{\frac{3}{2}}\;p^2}G^{3,1}_{1,3}\left(\frac{p^4}{64}\left|\begin{array}{c}1\\ \frac{1}{4},\frac{3}{4},1\end{array}\right.\right)\tag{4.21}$$

where $\mathbf{C}_1(t)$ is Fresnel integral [1, p.300 (7.3.3)].

Case(21): Put $c=\frac{3}{2},A=1,B=2,a_1=\frac{1}{4},b_1=\frac{1}{2},b_2=\frac{5}{4},y=-\frac{1}{4},k=2,$ in equation (3.1), we have

$$\mathfrak{L}\left\{\mathbf{C}_2(t):p\right\}=\frac{1}{2\;\pi^{\frac{1}{2}}\;p^{\frac{3}{2}}}G^{2,1}_{2,2}\left(p^2\left|\begin{array}{c}1,\frac{1}{2}\\ \frac{1}{4},\frac{3}{4}\end{array}\right.\right)$$

$$=\frac{1}{\sqrt{2}\;p^{\frac{3}{2}}}\;{}_2F_1\left[\begin{array}{c}\frac{1}{4},\frac{3}{4};\\ \frac{1}{2}\quad;\end{array}-\frac{1}{p^2}\right]\tag{4.22}$$

where $\mathbf{C}_2(t)$ is Fresnel integral [1, p.300 (7.3.3)].

Case(22): Put $c=2,A=1,B=2,a_1=\frac{1}{2},b_1=\frac{3}{2},b_2=\frac{3}{2},y=-\frac{1}{4},k=2,$ in equation (3.1),

we have

$$\mathfrak{L}\left\{S_i(t):p\right\}=\frac{1}{2\;p^2}G^{2,1}_{2,2}\left(p^2\left|\begin{array}{c}1,\frac{3}{2}\\ \frac{1}{2},1\end{array}\right.\right)$$

$$=\frac{1}{p^2}\;{}_2F_1\left[\begin{array}{c}\frac{1}{2},1;\\ \frac{3}{2}\quad;\end{array}-\frac{1}{p^2}\right]\tag{4.23}$$

where $S_i(t)$ is Sine integral [15, p.886 (8.230); 21, p.347].

Case(23): Put $c=2,A=1,B=2,a_1=\frac{1}{2},b_1=\frac{3}{2},b_2=\frac{3}{2},y=\frac{1}{4},k=2,$ in equation (3.1), we have

$$\mathfrak{L}\left\{Shi(t):p\right\}=\frac{1}{2\;p^2}G^{2,1}_{2,2}\left(-p^2\left|\begin{array}{c}1,\frac{3}{2}\\ \frac{1}{2},1\end{array}\right.\right)$$

$$=\frac{1}{p^2}\;{}_2F_1\left[\begin{array}{c}\frac{1}{2},1;\\ \frac{3}{2}\quad;\end{array}\frac{1}{p^2}\right]\tag{4.24}$$

where $Shi(t)$ is Hyperbolic Sine integral [15, p.886 (8.221), 21, p.347].

Case(24): Put $c=2,A=q+1,B=q,a_1=a_2=\ldots=a_{q+1}=1,b_1=b_2=\cdots=b_q=2,y=1,k=1,$ in equation (3.1), we have

$$\mathfrak{L}\left\{\mathbf{Li}_q(t):p\right\}=\frac{1}{p^2}G^{q+1,1}_{q,q+1}\left(-p\left|\begin{array}{c}1,\overbrace{2\;,\;2\;,\;\ldots,2}^{\text{q-1}}\\ \underbrace{1\;,\;1\;,\;\ldots,1}_{\text{q+1}}\end{array}\right.\right)\tag{4.25}$$

Here $\overbrace{2\ ,\ 2\ ,\ \ldots,2}^{\text{q-1}}$ denotes the numerator parameter "2" is written "$q-1$" times and $\underbrace{1\ ,\ 1\ ,\ \ldots,1}_{\text{q+1}}$ denotes the denominator parameter "1" is written "$q+1$" times and $\mathbf{Li}_q(t)$ is Polylogarithm function defined by $\mathbf{Li}_q(t)=\sum_{k=1}^{\infty}\frac{t^k}{k^q}$, (where $|t|<1$ and $q=2,3,4,\ldots$; when $q=2$ it is called Dilogarithm function);(See also 11, p.30 (1.11.14)).

Case(25): Put $c=\nu+2, A=1, B=2, a_1=1, b_1=\frac{3}{2}, b_2=\nu+\frac{3}{2}, y=-\frac{1}{4}, k=2$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{H}_\nu(t):p\}=\frac{1}{p^{\nu+2}\sqrt{\pi}}G^{3,1}_{3,3}\left(p^2\left|\begin{array}{c}1,\frac{3}{2},\nu+\frac{3}{2}\\ 1,\frac{\nu+2}{2},\frac{\nu+3}{2}\end{array}\right.\right)$$

$$=\frac{\Gamma(\nu+2)}{p^{\nu+2}\ \sqrt{\pi}\ 2^\nu\ \Gamma(\nu+\frac{3}{2})}\ {}_3F_2\left[\begin{array}{c}1,\Delta(2;\nu+2);\\ \frac{3}{2},\quad \nu+\frac{3}{2}\ ;\end{array}-\frac{1}{p^2}\right] \tag{4.26}$$

where $\mathbf{H}_\nu(t)$ is Struve function [1, p.496 (12.1.3); 19, p.217 (6.2.9.3)] and $\nu+2,\nu+\frac{3}{2}\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(26): Put $c=\nu+2, A=1, B=2, a_1=1, b_1=\frac{3}{2}, b_2=\nu+\frac{3}{2}, y=\frac{1}{4}, k=2$, in equation (3.1), we have

$$\mathfrak{L}\{\mathbf{L}_\nu(t):p\}=\frac{1}{p^{\nu+2}\sqrt{\pi}}G^{3,1}_{3,3}\left(-p^2\left|\begin{array}{c}1,\frac{3}{2},\nu+\frac{3}{2}\\ 1,\frac{\nu+2}{2},\frac{\nu+3}{2}\end{array}\right.\right)$$

$$=\frac{\Gamma(\nu+2)}{p^{\nu+2}\ \sqrt{\pi}\ 2^\nu\ \Gamma(\nu+\frac{3}{2})}\ {}_3F_2\left[\begin{array}{c}1,\Delta(2;\nu+2);\\ \frac{3}{2},\quad \nu+\frac{3}{2}\ ;\end{array}\frac{1}{p^2}\right] \tag{4.27}$$

where $\mathbf{L}_\nu(t)$ is Modified Struve function [1, p.498 (12.2.1); 19, p.217 (6.2.9.5)] and $\nu+2,\nu+\frac{3}{2}\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(27): Put $c=\mu+2, A=1, B=2, a_1=1, b_1=\frac{\mu-\nu+3}{2}, b_2=\frac{\mu+\nu+3}{2}, y=-\frac{1}{4}, k=2$, in equation (3.1), we have

$$\mathfrak{L}\{\mathrm{s}_{\mu,\nu}(t):p\}=\frac{2^{\mu-1}}{p^{\mu+2}\sqrt{\pi}}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)G^{3,1}_{3,3}\left(p^2\left|\begin{array}{c}1,\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2}\\ 1,\frac{\mu+2}{2},\frac{\mu+3}{2}\end{array}\right.\right)$$

$$=\frac{\Gamma(\mu+2)}{p^{\mu+2}\ [(\mu+1)^2-\nu^2]}\ {}_3F_2\left[\begin{array}{c}1,\Delta(2;\mu+2);\\ \frac{\mu-\nu+3}{2},\quad \frac{\mu+\nu+3}{2}\ ;\end{array}-\frac{1}{p^2}\right] \tag{4.28}$$

where $\mathrm{s}_{\mu,\nu}(t)$ is Lommel function [19, p.217 (6.2.9.1)] and $\frac{\mu-\nu+1}{2},\frac{\mu+\nu+1}{2},\mu+2\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(28): Put $c=1, A=0, B=3, b_1=\frac{1}{2}, b_2=\frac{1}{2}, b_3=1, y=-\frac{1}{256}, k=4$, in equation (3.1), we have

$$\mathfrak{L}\{ber(t):p\}=\frac{1}{p\ \sqrt{2\pi}}G^{2,1}_{2,2}\left(p^4\left|\begin{array}{c}1,\frac{1}{2}\\ \frac{1}{4},\frac{3}{4}\end{array}\right.\right)$$

$$= \frac{1}{p} \; {}_2F_1 \left[\begin{matrix} \frac{1}{4}, \frac{3}{4}; \\ \frac{1}{2} \quad ; \end{matrix} \; -\frac{1}{p^4} \right] \tag{4.29}$$

where $ber(t)$ is Kelvin's function [15, p.944 (8.564.1)].

Case(29): Put $c = 3, A = 0, B = 3, b_1 = \frac{3}{2}, b_2 = \frac{3}{2}, b_3 = 1, y = -\frac{1}{256}, k = 4$, in equation (3.1), we have

$$\mathfrak{L}\{bei(t) : p\} = \frac{1}{p^3 \sqrt{2\pi}} G_{2,2}^{2,1} \left(p^4 \middle| \begin{matrix} 1, \frac{3}{2} \\ \frac{3}{4}, \frac{5}{4} \end{matrix} \right)$$

$$= \frac{1}{2 \; \pi \; p^3} \; {}_2F_1 \left[\begin{matrix} \frac{3}{4}, \frac{5}{4}; \\ \frac{3}{2} \quad ; \end{matrix} \; -\frac{1}{p^4} \right] \tag{4.30}$$

where $bei(t)$ is Kelvin's function [15 p.944 (8.564.2)].

Case(30): Put $c = 1, A = q+1, B = q, a_1 = 1, a_2 = a_3 = \ldots = a_{q+1} = a, b_1 = b_2 = \cdots = b_q =$

$a+1, y = 1, k = 1$, in equation (3.1), we have

$$\mathfrak{L}\{\Phi(t, q, a) : p\} = \frac{1}{p} G_{1+q,q+2}^{q+2,1} \left(-p \middle| \begin{matrix} 1, \overbrace{a+1, a+1, \ldots, a+1}^{q} \\ 1, \underbrace{a, a, \ldots, a}_{q}, 1 \end{matrix} \right) \tag{4.31}$$

Here $\overbrace{a+1, a+1, \ldots, a+1}^{q}$ denotes the numerator parameter "$a+1$" is written "q" times and $\underbrace{a, a, \ldots, a}_{q}$ denotes the denominator parameter "a" is written "q" times; q is positive integer and $\Phi(t, q, a)$ is Lerch's transcendent given in the monographs [15, p.1039; 21, p.32 (1.6)].

We conclude our present investigation by observing that several other consequences of Laplace transforms can also be deduced in an analogous manner.

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