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A Two-Parameter Quasi-Lindley Mixture of Generalised Poisson Distribution

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Abstract:*A two-parameter Quasi-Lindley mixture of generalised Poisson distribution has been obtained by mixing the generalized Poisson distribution of Consul and Jain (1973) with the two-parameter Quasi-Lindley distribution of B.K.Sah (2015a). The first four moments about origin has been obtained. The estimation of parameters has been discussed by using the first two moments about origin and the probability mass function at x=0.The proposed distribution is a generalisation of Shankaran's (1970) Poisson- Lindley distribution and "A two-parameter Quasi-Lindley mixture of Poisson distribution" of B.K.Sah (2015). This distribution has been fitted to some data sets and it has been found that it provides closer fits than the Poisson-Lindley distribution of Sankaran and two-parameter Quasi-Lindley mixture of Poisson distribution of B.K.Sah (2015b).*
Keywords*:Lindley distribution, Generalised Poisson distribution, Compounding, Moments, Poisson- Lindley distribution, Two-Parameter Quasi-Lindley distribution, goodness of fit.*

I. INTRODUCTION

Lindley (1958) introduced a one-parameter continuous distribution given by its probability density function,

$$f(x,\phi) = \frac{\phi^2}{\phi+1}(1+x)e^{-\phi x} \quad ; \quad x>0,\ \phi>0 \tag{1.1}$$

Ghitaney et al (2008) studied the various properties of this distribution.

Shankaran (1970) obtained a Poisson-Lindley distribution (PLD) given by its probability mass function

$$P(x) = \frac{\phi^2(\phi+2+x)}{(\phi+1)^{x+3}} \ ;x=0,1,2,3, \tag{1.2}$$

assuming that the parameter of a Poisson distribution has Lindley distribution which can symbolically be expressed as

$$\text{Poisson}(m) \underset{m}{\wedge} \text{Lindley} \ (\phi) . \tag{1.3}$$

He discussed that this distribution has applications in errors and accidents. The first four moments about origin of this distribution have been given by

$$\mu_1^{/} = \frac{(\phi+2)}{\phi(\phi+1)} \tag{1.5}$$

$$\mu_2^{/} = \frac{(\phi+2)}{\phi(\phi+1)} + \frac{2(\phi+3)}{\phi^2(\phi+1)} \tag{1.6}$$

$$\mu_3^{/} = \frac{(\phi+2)}{\phi(\phi+2)} + \frac{6(\phi+3)}{\phi^2(\phi+1)} + \frac{6(\phi+4)}{\phi^3(\phi+1)} \tag{1.7}$$

$$\mu_4^{/} = \frac{(\phi+2)}{\phi(\phi+1)} + \frac{14(\phi+3)}{\phi^2(\phi+1)} + \frac{36(\phi+4)}{\phi^3(\phi+1)} + \frac{36(\phi+5)}{\phi^4(\phi+1)} \tag{1.8}$$

and its variance as

$$\mu_2 = \left(\phi^3+4\phi^2+6\phi+2\right) \Big/ \phi^2(\phi+1)^2 \tag{1.9}$$

Ghitany et al (2009) discussed the estimation methods for the one parameter Poisson-Lindley distribution (1.2) and its applications.

A two-parameter Quasi-Lindley distribution ((QLD) of B.K.Sah (2015a) with parameters α and ϕ is defined by its probability density function (pdf)

$$f(x;\phi,\alpha)=\frac{\phi^{3}(1+x+\alpha x^{2})e^{-\phi x}}{(\phi^{2}+\phi+2\alpha)};\phi>0,x>0,(\phi^{2}+\phi+2\alpha)>0 \qquad (1.10)$$

It can easily be seen that at α =0 it reduces to the Lindley distribution.

A two parameter Quasi-Lindley mixture of Poisson distribution of B.K.Sah(2015b) has been obtained by mixing Poisson distribution with the Quasi- Lindley distribution (2.1). Its probability mass function has been given by

$$\text{QPLD}(x;\phi,\alpha)=\frac{\phi^{3}}{(\phi^{2}+\phi+2\alpha)}\left[\frac{(1+\phi)^{2}+(x+1)(1+\phi)+\alpha(x+1)(x+2)}{(1+\phi)^{x+3}}\right] \qquad (1.11)$$

It is also called Quasi Poisson –Lindley distribution(QPLD).It reduces to the Poisson Lindley distribution of Shankaran (1970) at α = 0

Consul and Jain (1973) obtained a two-parameter generalized Poisson distribution (GPD) given by its probability function

$$P(x)=\frac{\lambda(\lambda+x\theta)^{x-1}e^{-(\lambda+x\theta)}}{x!} \qquad \ldots \quad \ldots \quad \ldots \qquad (1.12)$$

Where x = 0, 1, 2, … ; $\lambda>0;|\theta|<1$

In this paper, a generalization of the two-parameter Quasi-Lindley mixture of Poisson distribution (1.11) has been obtained by mixing the GPD (1.12) with the QLD (1.10).

II. A TWO-PARAMETER QUASI - LINDLEY MIXTURE OF GPD

Suppose that the parameter λ in the GPD (1.10) is a random variable and follows the two-parameter Quasi Lindley distribution (1.10) with parameter ϕ *and* α .We have thus the Quasi- Lindley mixture of the GPD is obtained as

$$P(x;\alpha,\phi,\theta)=\int_0^\infty\frac{\lambda(\lambda+\theta x)^{x-1}e^{-(\lambda+x\theta)}}{\Gamma(x+1)}\cdot\frac{\phi^{3}(1+x+\alpha x^{2})e^{-\phi x}}{(\phi^{2}+\phi+2\alpha)}\,dx \qquad (2.2)$$

$$=\int_0^\infty\frac{\lambda(\lambda+\theta x)^{x-1}e^{-(\lambda+x\theta)}}{\Gamma(x+1)}\cdot\frac{\phi^{3}(1+\lambda+\alpha\lambda^{2})e^{-\phi\lambda}}{(\phi^{2}+\phi+2\alpha)}\,d\lambda$$

$$=\frac{\phi^{3}e^{-\theta x}}{\Gamma(x+1)(\phi^{2}+\phi+2\alpha)}\int_0^\infty\lambda^{x}\left(1+\frac{\theta x}{\lambda}\right)^{x-1}(1+\lambda+\alpha\lambda^{2})e^{-\lambda(1+\phi)}\,d\lambda$$

After a little simplification, we get

$$=\frac{\phi^{3}e^{-x\theta}}{(\phi^{2}+\phi+2\alpha)\,x!}\sum_{i=0}^{x-1}\frac{(x-1)!(\theta x)^{i}}{i!(x-i-1)!}\left[\frac{\Gamma(x-i+1)}{(1+\phi)^{x-i+1}}+\frac{\Gamma(x-i+2)}{(1+\phi)^{x-i+2}}+\frac{\alpha\,\Gamma(x-i+3)}{(1+\phi)^{x-i+3}}\right]$$

After a little simplification and arrangement of terms, we get

$$P(x;\alpha,\phi,\theta)=\frac{\phi^{3}e^{-\theta x}}{(\phi^{2}+\phi+2\alpha)(1+\phi)^{x+3}}\left[\begin{array}{l}\left\{(1+\phi)^{2}+(1+\phi)(1+x)+\alpha(2+x)(1+x)\right\}+\\ \displaystyle\sum_{i=1}^{x-1}\frac{\theta^{i}x^{i-1}(x-i)\left\{(1+\phi)^{2}+(1+\phi)(x-i+1)+\alpha(x-i+2)(x-i+1)\right\}}{i!(1+\phi)^{-i}}\end{array}\right] \qquad (2.3)$$

The expression (2.3) is the pmf of two-parameter Quasi-Lindley mixture of GPD (QLMGPD).It can be seen that at $\theta=0$ *, it reduces to the QPLD (1.11) and hence it may be termed as 'generalized Quasi Poisson-Lindley distribution' (GQPLD). It can also be noted that P(X=0) in the GQPLD becomes independent of* θ *, the additional parameter introduced in the distribution and its role starts from P(X=1) onwards. It can also be seen that at* $\theta=0$ *and* $\alpha=0$ *it reduces to the PLD of Sankaran (1970). At* $\alpha=0$ *, it reduces to the generalized Poisson-Lindley distribution (GPLD) of B.K.Sah (2013).*

III. MOMENTS

The rth moment about origin of the GQPLD (2.3) can be obtained as

$$\mu_r^{/} = E\left[E(X^r/\lambda)\right] \tag{3.1}$$

From (2.2) we thus get

$$\mu_r^{/} = \int_0^{\infty}\left[\sum_{x=0}^{\infty}\frac{x^r\,\lambda\left(\lambda+x\theta\right)^{x-1}e^{-(\lambda+\theta x)}}{\Gamma(x+1)}\right]\frac{\phi^3\left(1+\lambda+\alpha\lambda^2\right)e^{-\phi\lambda}}{\left(\phi^2+\phi+2\alpha\right)}\,d\lambda \tag{3.2}$$

Obviously the expression under bracket is the rth moment about origin of the GPD (1.12).
Taking r = 1 in (3.2) and using the mean of the GPD), the mean of the GPLD is obtained as

$$\mu_1^{/} = \frac{\phi^3}{(\phi^2+\phi+2\alpha)}\int_0^{\infty}\frac{\lambda}{(1-\theta)}\,.\,(1+\lambda+\alpha\lambda^2)e^{-\phi\lambda}\,d\lambda$$

$$= \frac{(\phi^2+2\phi+6\alpha)}{\phi\left(\phi^2+\phi+2\alpha\right)(1-\theta)} \tag{3.3}$$

Taking r = 2 in (3.2) and using the second moment about origin of the GPD, the second moment about origin of the GQPLD is obtained as

$$\mu_2^{/} = \frac{\phi^3}{(\phi^2+\phi+2\alpha)}\int_0^{\infty}\left[\frac{\lambda}{(1-\theta)^3}+\frac{\lambda^2}{(1-\theta)^2}\right](1+\lambda+\alpha\lambda^2)e^{-\phi\lambda}\,d\lambda$$

which after a little simplification gives

$$= \frac{(\phi^2+2\phi+6\alpha)}{\phi\left(\phi^2+\phi+2\alpha\right)(1-\theta)^3}+\frac{(2\phi^2+6\phi+24\alpha)}{\phi^2\left(\phi^2+\phi+2\alpha\right)(1-\theta)^2} \tag{3.4}$$

Taking r = 3 in (3.2) and using the third moment about origin of the GPD ,the third moment about origin of the GQPLD is obtained as

$$\mu_3^{/} = \frac{\phi^3}{(\phi^2+\phi+2\alpha)}\int_0^{\infty}\left[\frac{\lambda(1+2\theta)}{(1-\theta)^5}+\frac{3\lambda^2}{(1-\theta)^4}+\frac{\lambda^3}{(1-\theta)^3}\right](1+\lambda+\alpha\lambda^2)e^{-\phi\lambda}\,d\lambda$$

which after a little simplification gives

$$= \frac{(1+2\theta)(\phi^2+2\phi+6\alpha)}{\phi\left(\phi^2+\phi+2\alpha\right)(1-\theta)^5}+\frac{3(2\phi^2+6\phi+24\alpha)}{\phi^2\left(\phi^2+\phi+2\alpha\right)(1-\theta)^4}+\frac{6(\phi^2+4\phi+20\alpha)}{\phi^3\left(\phi^2+\phi+2\alpha\right)(1-\theta)^3} \tag{3.5}$$

Taking r = 4 in (3.2) and using the fourth moment about origin of the GPD ,the fourth moment about origin of the GQPLD is obtained as

$$\mu_4^{/} = \frac{\phi^3}{(\phi^2+\phi+2\alpha)}\int_0^{\infty}\left[\frac{\left(1+8\theta+6\theta^2\right)\lambda}{(1-\theta)^7}+\frac{(7+8\theta)\lambda^2}{(1-\theta)^6}+\frac{6\lambda^3}{(1-\theta)^5}+\frac{\lambda^4}{(1-\theta)^4}\right](1+\lambda+\alpha\lambda^2)e^{-\phi\lambda}\,d\lambda$$

which after a little simplification gives

$$= \frac{(1+8\theta+6\theta^2)(\phi^2+2\phi+6\alpha)}{\phi\left(\phi^2+\phi+2\alpha\right)(1-\theta)^7}+\frac{(7+8\theta)(2\phi^2+6\phi+24\alpha)}{\phi^2\left(\phi^2+\phi+2\alpha\right)(1-\theta)^6}+\frac{6(6\phi^2+24\phi+120\alpha)}{\phi^3\left(\phi^2+\phi+2\alpha\right)(1-\theta)^5}+$$

$$\frac{(24\phi^2+120\phi+720\alpha)}{\phi^4\left(\phi^2+\phi+2\alpha\right)(1-\theta)^4} \tag{3.6}$$

It can easily be seen that at $\theta = 0$, these moments reduce to the respective moments of the two parameter Quasi Poisson-Lindley distribution(1.11) and It can also be seen that at $\theta = 0$ *and* $\alpha = 0$ it reduces to the respective moments of PLD of Sankaran (1970). At $\alpha = 0$, it reduces to the respective moments of the generalized Poisson-Lindley distribution (GPLD) of B.K.Sah (2013)

IV. ESTIMATION OF PARAMETERS

The GQPLD have three parameters ϕ , α and θ . Here, we have obtained the estimates of these parameters by using P(x=0) and the first two moments about origin.

$$P(X=0)=\frac{\phi^3(\phi^2+3\phi+2+2\alpha)}{(1+\phi)^3(\phi^2+\phi+2\alpha)}=k\,(say) \tag{4.1}$$

Or, $$k=\frac{2\phi^3(1+\phi)}{(1+\phi)^3(\phi^2+\phi+2\alpha)}+\frac{\phi^3(\phi^2+\phi+2\alpha)}{(1+\phi)^3(\phi^2+\phi+2\alpha)}$$

After a little implication, we get an estimate of α

$$\alpha=\frac{\phi(1+\phi)\left[2\phi^2-\left\{k(1+\phi)^3-\phi^3\right\}\right]}{2\left\{k(1+\phi)^3-\phi^3\right\}} \quad \ldots \quad \ldots \tag{4.2}$$

Replacing the first population moment by respective sample moment and putting the value of α in expression (3.2), an estimate of θ can be obtained as

$$(1-\theta)=\frac{(\phi^2+2\phi+6\alpha)}{\mu_1'\phi(\phi^2+\phi+2\alpha)} \quad \ldots \quad \ldots \tag{4.3}$$

Replacing the second population moment by respective sample moment and putting the values of α *and* $(1-\theta)$ in the expression of μ_2' (3.3), we get an estimate of ϕ

$$k_1\left[6\phi^2(1+\phi)-(2\phi+1)\left\{k(1+\phi)^3-\phi^3\right\}\right]^2 - 2\phi(1+\phi)\left[2\mu_1'\phi^6+2\mu_1'\phi^5+24\phi^4+24\phi^3-(10\phi^2+6\phi)\left\{k(1+\phi)^3-\phi^3\right\}\right]=0 \tag{4.4}$$

Where $k_1=\dfrac{\mu_2'}{(\mu_1')^2}$.

The expression (4.4) is the Polynomial equation in ϕ which may be solved by using the Newton-Raphson's method or Regula-Falsi method.

V. GOODNESS OF FIT

The two-parameter Quasi-Lindley mixture of generalized Poisson distribution can also be called generalised Quasi Poisson-Lindley distribution (QPLD). It has been fitted to a number of discrete data-sets to which earlier the Poisson-Lindley distribution has been fitted. Here the fitting of the three-parameter GQPLD has been presented in the following table. The data is the Student's historic data Hemocytometer counts of yeast cell, used by Borah (1984) for fitting the Gegenbauer distribution

Table -I

Hemocytometer Counts of Yeast Cell

Number of Yeast Cell per square	Observed frequency	Expected frequency of PLD	Expected frequency of two-parameter PLD	Expected frequency of QPLD	Expected frequency of GQPLD
0	213	234.4	227.6	224.8	213.0
1	128	99.3	101.5	106.0	126.6
2	37	40.4	43.6	45.1	40.9
3	18	16.0	17.9	17.5	14.5
4	3	6.2	6.8	5.9	3.7
5	1	3.7	2.8	0.7	1.3
Total	400	400.0	400.0	400.0	400.0

$\mu_1^{/}$	0.6825				
$\mu_2^{/}$	1.2775				
$\hat{\phi}$		1.9602	3.6728	1.367243	1.405885
$\hat{\alpha}$			-0.0916	-0.38101314	-0.2494
$\hat{\theta}$					-0.189754
χ^2		14.3	12.25	7.66	0.78
$d.f$		4	3	3	2
$P-value$		<0.005	<0.01	>0.025	0.67

VI. CONCLUSION

It has been observed that the proposed GQPLD (2.3) gives better fit to the above data-set than PLD of Sankaran(1970), a two-parameter PLD of Rama Sanker and A.Mishra(2013), and QPLD(1.11) of B.K.Sah(2015b).

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