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ON QUASI-TERNARY Γ-IDEALS AND BI-TERNARY Γ-IDEALS IN TERNARY Γ-SEMIRINGS

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***Abstract* :** We introduce the notions of quasi-ternary Γ-ideal and bi-ternary Γ-ideal in ternary Γ-semirings and study some properties of these two ternary Γ-ideals. We also characterize regular ternary Γ-semiring in terms of these two subsystems of ternary Γ-semirings.

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***Key Words* :**ternary Γ-semiring, quasi-ternary Γ-ideals, bi-ternary Γ-ideals, regular ternary Γ-semiring.

I.INTRODUCTION

Good and Hughes [3] introduced the notion of bi-ideal and Steinfeld [5] introduced the notion of quasi-ideal. Sioson [4] studied some properties of quasi-ideals of ternary semigroups. In [1], Dixit and Dewan studied about the quasi-ideals and bi-ideals of ternary semigroups. Quasi-ideals are generalization of right ideals, lateral ideals, and left ideals whereas bi-ideals are generalization of quasi-ideals. In [2], we introduced the notion of ternary semiring. Syam Julius Rajendra , Madhusudhana Rao and Sajani Lavanya [6], introduced the completely regular ternary Γ-ideals in partially ordered ternary Γ-semiring. Our main purpose of this paper is to introduce the notions of quasi-ternary Γ-ideal and bi-ternary Γ-ideal in ternary Γ-semirings and study regular ternary Γ-semiring in terms of these two subsystems of ternary Γ-semirings.

II.PRELIMINARIES

Definition* 2.1:** Let T and Γ be two additive commutative semigroups. T is said to be a ***Ternary Γ-semiring if there exist a mapping from T ×Γ× T ×Γ× T to T which maps $(x_1, \alpha, x_2, \beta, x_3) \rightarrow [x_1\alpha x_2\beta x_3]$ satisfying the conditions :

i) $[[a\alpha b\beta c]\gamma d\delta e] = [a\alpha[b\beta c\gamma d]\delta e] = [a\alpha b\beta[c\gamma d\delta e]]$
ii)$[(a + b)\alpha c\beta d] = [a\alpha c\beta d] + [b\alpha c\beta d]$
iii) $[a\alpha\ (b + c)\beta d] = [a\alpha b\beta d] + [a\alpha c\beta d]$
iv) $[a\alpha b\beta(c + d)] = [a\alpha b\beta c] + [a\alpha b\beta d]$ for all $a, b, c, d \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Definition* 2.2:** An element 0of a ternary Γ-semiring T is said to be an ***absorbingzero of T provided $0 + x = x = x + 0$and $0\alpha a\beta b = a\alpha 0\beta b = a\alpha b\beta 0 = 0\ \forall\ a, b, x \in$ T and $\alpha, \beta \in \Gamma$.

***Note* 2.3 :**Throughout this paper, T will always denote a ternary Γ-semiring with zero and unless otherwise stated a ternary Γ-semiring means a ternary Γ-semiring with zero.

Definition* 2.4**: Let T be ternary Γ-semiring. A non empty subset ‘S’ is said to be a ***ternary subΓ-semiring of T if S is an additive subsemigroup of T and $a\alpha b\beta c \in$ S for all $a,b,c \in$ S and $\alpha, \beta \in \Gamma$.

***Note* 2.5 :** A non empty subset S of a ternary Γ-semiring T is a ternary subΓ-semiring if and only if S + S ⊆ S and SΓSΓS ⊆ S.

III. TERNARY Γ-IDEAL

Definition* 3.1 :**A ternary Γ-semiring T is said to be ***zero divisor free (ZDF) if for $a, b, c \in T, \alpha, \beta \in \Gamma$, $[a\alpha b\beta c] = 0$ implies that $a = 0$ or $b = 0$ or $c = 0$.
Definition* 3.2 :**A ternary Γ-semiring T is said to be ***multiplicatively left Γ-cancellative (MLC) if $a\Gamma b\Gamma x = a\Gamma b\Gamma y$ implies that $x = y$ for all $a, b, x, y \in$ T.

Definition* 3.3:** A ternary Γ-semiring T is said to be ***multiplicatively laterally Γ-cancellative (MLLC) if $a\Gamma x\Gamma b = a\Gamma y\Gamma b$ implies that $x = y$ for all $a, b, x, y \in$ T.

Definition* 3.4 :**A ternary Γ-semiring T is said to be ***multiplicatively right* Γ*-cancellative (MRC) if $x\Gamma a\Gamma b = y\Gamma a\Gamma b$ implies that $x = y$ for all $a, b, x, y \in$ T.

Definition* 3.5 :**A ternary Γ-semiring T is said to be ***multiplicatively* Γ*-cancellative (MC) if it is multiplicative left Γ-cancellative (MLC), multiplicative right Γ-cancellative (MRC) and multiplicative laterally Γ-cancellative (MLLC).

***Theorem* 3.6: A multiplicative Γ-cancellative ternary Γ-semiring T is zero divisor free**.
Proof: Let T be a multiplicative Γ-cancellative ternary Γ-semiring and $a\Gamma b\Gamma c= 0$ for a; b; $c\in$T.Suppose $b \neq 0$ and $c\neq 0$. Then by right Γ-cancellativity, $a\Gamma b\Gamma c= 0 = 0\Gamma b\Gamma c$implies that $a= 0$. Similarly, we can show that $b = 0$ if $a \neq 0$ and $c \neq 0$ or $c = 0$ if $a \neq 0$ and $b \neq 0$. Consequently, T is zero divisor free.

Definition* 3.9** : A nonempty subset A of a ternary Γ-semiring T is said to be ***left ternary* Γ*-ideal of T if (1) $a, b \in$ A implies $a + b \in$ A. (2) $b, c \in$ T, $a \in$ A, $\alpha, \beta\in\Gamma$ implies $b\alpha c\beta a \in$ A.

***Note* 3.10** : A nonempty subset A of a ternary Γ-semiring T is a left ternaryΓ-ideal of T if and only if A is additive subsemigroup of T and TΓTΓA $\subseteq$ A.

Definition* 3.11** : A nonempty subset of a ternary Γ-semiring T is said to be a ***lateral ternary* Γ*-ideal of T if (1) $a, b \in$ A $\Rightarrow a + b \in$ A. (2) $b, c\in$ T, $a \in$ A, $\alpha, \beta\in\Gamma\Rightarrow b\alpha a\beta c \in$ A.

***Note* 3.12**: A nonempty subset of A of a ternary Γ-semiring T is a lateral ternary Γ-ideal of T if and only if A is additive subsemigroup of T and TΓAΓT $\subseteq$ A.
Definition* 3.13** : A nonempty subset A of a ternary Γ-semiring T is a ***right ternary* Γ*-ideal of T if (1) $a, b \in$ A $\Rightarrow a + b \in$ A. (2) $b, c \in$ T , $a \in$ A, $\alpha, \beta\in\Gamma\Rightarrow a\alpha b\beta c \in$ A.

***Note* 3.14**: A nonempty subset A of a ternary Γ-semiring T is a right ternary Γ-ideal of T if and only if A is additive subsemigroup of T and AΓTΓT $\subseteq$ A.

Definition* 3.15**: A nonempty subset A of a ternary Γ-semiring T is said to be ***ternary* Γ*-idealof T if
(1) $a, b \in$ A $\Rightarrow a + b \in$ A
(2) $b, c \in$ T , $a \in$ A, $\alpha, \beta\in\Gamma\Rightarrow b\alpha c\beta a\in$ A, $b\alpha a\beta c \in$ A, $a\alpha b\beta c \in$ A.

***Note* 3.16** : A nonempty subset A of a ternary Γ-semiring T is a ternaryΓ-ideal of T if and only if it is left ternaryΓ-ideal, lateral ternaryΓ-ideal and right ternaryΓ-ideal of T.

Definition* 3.17**: A ternary Γ-ideal A of a ternary Γ-semiring T is said to be a ***proper ternary* Γ*-ideal of T if A is different from T.

Definition* 3.18**: A left ternary Γ-ideal A of a ternary Γ-semiring T is said to be the ***principal left ternary* Γ*-ideal generated by a if A is a left ternary Γ-ideal generated by $\{a\}$ for some $a\in$ T. It is denoted by L (a) or $<a>_l$.

***Theorem* 3.19 : If T is a ternary Γ-semiring and *a* $\in$ T then**
$<a>_l=$ $\left\{\sum_{i=1}^{n} r_i\alpha_i t_i\beta_i a + na : r_i, t_i \in T, \alpha_i, \beta_i \in \Gamma \text{ and } n \in z_0^+\right\}$**. Where Σ denotes a finite sum and** z_0^+ **is the set of all positive integer with zero.**

***Proof* :**Let A = $\left\{\sum_{i=1}^{n} r_i\alpha_i t_i\beta_i a + na : r_i, t_i \in T, \alpha_i, \beta_i \in \Gamma \text{ and } n \in z_0^+\right\}$. Let $a, b\in$ A.

$a, b\in$ A . $a = \sum r_i\alpha_i t_i\beta_i a + na$ and $b = \sum r_j\alpha_j t_j\beta_j a + na$ for r_i, t_i, r_j, $t_j\in$ T, $\alpha_i, \beta_i, \alpha_j, \beta_j \in \Gamma$ and $n\in z_0^+$. Now $a + b = \sum r_i\alpha_i t_i\beta_i a + na + \sum r_j\alpha_j t_j\beta_j a + na \Rightarrow a + b$ is a finite sum.

Therefore $a + b\in$ A and hence A is additive subsemigroup of T. For $t_1,t_2\in$ T and $a\in$ A.

Then $t_1\alpha t_2\beta a = t_1\alpha t_2(\sum r_i\alpha_i t_i\beta_i a + na) = \sum r_i\alpha_i t_i\beta_i(t_1\alpha t_2\beta a) + n(t_1\alpha t_2\beta a) \in$ A

Therefore $t_1\alpha t_2\beta a \in$A and hence A is a left ternary Γ-ideal of T.

Let L be a left ternary Γ-ideal of T containing a.

Let $r\in$ A**.** Then $r = \sum r_i\alpha_i t_i\beta_i a + na$ for $r_i, t_i \in T$, $\alpha_i, \beta_i \in \Gamma, n \in z_0^+$.

If $r = \sum r_i\alpha_i t_i\beta_i{}_i a + na \in L$.

Therefore A ⊆L and hence A is a smallest left ternary Γ-ideal containing *a*.

Therefore A = L(*a*) = $\left\{ \sum_{i=1}^{n} r_i\alpha_i t_i\beta_i a + na : r_i, t_i \in T, \alpha_i, \beta_i \in \Gamma \text{ and } n \in z_0^+ \right\}$.

Note **3.20** : if T is ternary Γ-semiring and $a \in$ T then L(a) = $T^e \Gamma T^e \Gamma a + na$.

Definition **3.21** : A nonempty subset of a ternary Γ-semiring T is said to be a ***lateral ternary Γ-ideal*** of T if
(1) *a, b* ∈ A implies *a* + *b*∈ A.
(2) *b, c*∈ T , *α, β* ∈ Γ, *a*∈ A implies *bαaβc*∈ A.

Note **3.22**: A nonempty subset of A of a ternary Γ-semiring T is a lateral ternary Γ-ideal of T if and only if A is additive subsemigroup of T, TΓAΓT ⊆ A.

Definition **3.23** : A lateral ternary Γ-ideal A of a ternary Γ-semiring T is said to be the ***principal lateral ternary Γ-ideal generated by a*** if A is a lateral ternary Γ-ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by M (*a*) (or) $<a>_m$.

Theorem **3.24 : If T is a ternary Γ-semiring and *a*∈ T then**

$$< a >_m = \left\{ \begin{array}{l} \sum_{i=1}^{n} r_i\alpha_i a\beta_i t_i + \sum_{j=1}^{n} u_j\alpha_j v_j\beta_j a\gamma_j p_j\delta_j q_j + na : r_i, t_i, u_j v_j p_j q_j \in T, \\ \alpha_i, \beta_i, \alpha_j, \beta_j, \gamma_j, \delta_j \in \Gamma \text{ and } n \in z_0^+ \end{array} \right\},$$ **and Σ denotes a finite sum and z_0^+ is the set of all positive integer with zero.**

Definition **3.25** : A nonempty subset A of a ternary Γ-semiring T is a ***right ternary Γ-ideal***of T if
(1) *a, b* ∈ A implies *a* + *b*∈ A.
(2) *b, c* ∈ T , *α, β*∈ Γ, *a*∈ A implies *aαbβc*∈ A.

Note 3.26 : A nonempty subset A of a ternary Γ-semiring T is a right ternary Γ-ideal of T if and only if A is additive subsemigroup of T, AΓTΓT ⊆ A.

Definition **3.27** : A right ternary Γ-ideal A of a ternary Γ-semiring T is said to be a ***principal right ternary Γ-ideal generated by a*** if A is a right PO-ternary Γ-ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by R (*a*) (or) $<a>_r$.

Theorem **3.28 : If T is a ternary Γ-semiring and *a*∈ T then**

$<a>_r$ $= \left\{ \sum_{i=1}^{n} a\alpha_i r_i\beta_i t_i + na : r_i, t_i \in T, \alpha_i, \beta_i \in \Gamma \text{ and } n \in z_0^+ \right\}$ **, Σ denotes a finite sum and z_0^+ is the set of all positive integer with zero.**

Definition **3.29** : A nonempty subset A of a ternary Γ-semiring T is a ***two sided ternary Γ-ideal*** of T if

(1) *a, b* ∈ A implies *a* + *b*∈ A
(2) *b, c*∈ T ,*α, β*∈ Γ, *a*∈ A implies *bαcβa*∈ A, *aαbβc*∈ A.

Note **3.30**: A nonempty subset A of a ternary Γ-semiring T is a two sided ternary Γ-ideal of T if and only if it is both a left ternary Γ-ideal and a right ternary Γ-ideal of T.

Definition **3.31** : A two sided ternary Γ-ideal A of a ternary Γ-semiring T is said to be the ***principal two sided ternary Γ-ideal*** provided A is a two sided ternary Γ-ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by T (*a*) (or) $<a>_t$.

Theorem **3.32 : If T is a ternary Γ-semiring and *a*∈ T then**

$$< a >_t = \left\{ \begin{array}{l} \sum_{i=1}^{n} r_i\alpha_i s_i\beta_i a + \sum_{j=1}^{n} a\alpha_j t_j\beta_j u_j + \sum_{k=1}^{n} l_k\alpha_k m_k\beta_k a\gamma_k p_k\delta_k q_k + na : \\ r_i, s_i, t_j, u_j, l_k m_k, p_k, q_k \in T, \alpha_i, \beta_i, \alpha_j, \beta_j, \alpha_k, \beta_k, \gamma_k, \delta_k \in \Gamma \text{ and } n \in Z_0^+ \end{array} \right\}$$ **and Σ denotes a finite sum and z_0^+ is the set of all positive integer with zero.**

Definition* 3.33** : A nonempty subset A of a ternary Γ-semiring T is said to be ***ternary* Γ*-ideal of T if
(1) $a, b \in$ A implies $a + b \in$ A
(2) $b, c \in$ T, $\alpha, \beta \in \Gamma, a \in$ A implies $b\alpha c\beta a \in$ A, $b\alpha a\beta c \in$ A, $a\alpha b\beta c \in$ A.

***Note* 3.34** : A nonempty subset A of a ternary Γ-semiring T is a ternary Γ-ideal of T if and only if it is left ternary Γ-ideal, lateral ternary Γ-ideal and right ternary Γ-ideal of T.

Definition 3.35 : A ternary Γ-ideal A of a ternary Γ-semiring T is said to be ***a principal ternary* Γ*-ideal*** provided A is aternary Γ-ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by J (a) (or) <a>.

***Theorem* 3.36 : If T is a ternary Γ-semiring and $a \in$ T then**

$$<a>=\{ \sum_{i=1}^{n} p_i\alpha_i q_i\beta_i a + \sum_{j=1}^{n} a\alpha_j r_j\beta_j s_j + \sum_{k=1}^{n} t_k\alpha_k a\beta_k u_k + \sum_{l=1}^{n} v_l\alpha_l w_l\beta_l a\gamma_l x_l\delta_l y_l + na$$

$$: p_i, q_i, r_j, s_j, t_k, u_k, v_l, w_l, x_l y_l \in T, \alpha_i, \beta_i, \alpha_j, \beta_j, \alpha_k, \beta_k, \alpha_l, \beta_l, \gamma_l, \delta_l \in \Gamma, n \in Z_0^+ \}.$$

Where Σ denotes a finite sum and z_0^+ is the set of all positive integer with zero.

4.Quasi-ternary Γ-ideal and bi-ternary Γ-ideal in ternary Γ-semirings

***Definition* 4.1**: An additivesubsemigroup Q ofa ternaryΓ-semiring T is calleda quasi-ternary Γ-ideal of T if QΓTΓT∩(TΓQΓT+TΓTΓQΓTΓT)∩TΓTΓQ⊆Q.

***Note* 4.2**: Every quasi-ternary Γ-ideal of a ternary Γ-semiring T is a ternary Γ-sub semiring of T.

***Lemma* 4.3**:**Every left, right, and lateral ternary Γ-ideal of a ternary Γ-semiring T is a quasi-ternary Γ-ideal of T.**

***Proof*:** Assume that Q is a left ternary Γ-ideal of T. Then TΓTΓQ ⊆ Q, but TΓTΓQ ∩ (TΓQΓT ∪ TΓTΓQΓTΓT] ∩ QΓTΓT ⊆ TΓTΓQ ⊆ Q. Hence Q is a quasi-ternary Γ-ideal of T. Similarly we can prove that the remaining parts.

***Remark* 4.4:** The converse of Lemma 4.3 is not true, in general, that is, a quasi-ternary Γ-ideal may not be a left, a right, or a lateral ternary Γ-ideal of T. This follows from the following example.

***Example* 4.5:** Let T $= M_2(z_0^-)$ betheternaryΓ-semiringofthesetofall2×2squarematrices over Z_0^-, the set of all nonpositive integers and Γ be the set of all 2×2 square matrices over Z^-, the set of all negative integers. Let Q=$\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0^- \right\}$. Then we can easilyverify that Q is a quasi-ternary Γ-ideal of T, butQ is not a right ternary Γ-ideal, a lateral ternary Γ-ideal, or a left ternary Γ-ideal of T.

***Theorem* 4.6:If Q is a quasi-ternary Γ-ideal of a ternary Γ-semiring T and S is a ternary Γ-sub semiring of T, then Q∩S is a quasi-ternary Γ-ideal of S.**

***Proof*:** Assume that Q_1 = Q∩S ≠ ∅. Since Q_1⊆ Q, it follows that
SΓSΓQ_1 ∩ SΓQ_1ΓS ∩ Q_1ΓSΓS ⊆ TΓTΓQ ∩ TΓQΓT ∩ QΓTΓT ⊆ Q.
Since Q_1⊆ S and S is a ternary Γ-subsmigroup of T. We have SΓSΓQ_1 ∩ SΓQ_1ΓS ∩ Q_1ΓSΓS ⊆ S.
Then SΓSΓQ_1 ∩ SΓQ_1ΓS ∩ Q_1ΓSΓS ⊆Q_1.Therefore Q_1 is quasi-ternary Γ-ideal of S.

***Lemma* 4.7. The intersection of arbitrary collection of quasi-ternary Γ-ideals of a ternary Γ-semiring T is a quasi-ternary Γ-ideal of T.**

***Proof*:** Let $\{Q_\alpha\}_{\alpha \in \Delta}$ be a family of -ternary Γ-ideals of T and let Q = $\bigcap_{\alpha \in \Delta} Q_\alpha$

Assume that Q is not empty. Since Q_α is a quasi-ternary Γ-ideal for each $\alpha \in \Delta$.
Then Q_αΓTΓT ∩ (TΓQ_αΓT + TΓTΓQ_αΓTΓT) ∩ (TΓTΓQ_α]⊆ Q_α for each $\alpha \in \Delta$.

Now for each $\alpha \in \Delta$ $T\Gamma T\Gamma Q = T\Gamma T\Gamma(\bigcap_{\alpha\in\Delta} Q_\alpha) = \bigcap_{\alpha\in\Delta} T\Gamma T\Gamma Q_\alpha \subseteq T\Gamma T\Gamma Q_\alpha$,

$T\Gamma Q\Gamma T = T\Gamma(\bigcap_{\alpha\in\Delta} Q_\alpha)\Gamma T = \bigcap_{\alpha\in\Delta} T\Gamma Q_\alpha\Gamma T \subseteq T\Gamma Q_\alpha\Gamma T$,

$T\Gamma T\Gamma Q\Gamma T\Gamma T = T\Gamma T\Gamma(\bigcap_{\alpha\in\Delta} Q_\alpha)\Gamma T\Gamma T = \bigcap_{\alpha\in\Delta} T\Gamma T\Gamma Q_\alpha\Gamma T\Gamma T \subseteq T\Gamma T\Gamma Q_\alpha\Gamma T\Gamma T$, and

$T\Gamma T\Gamma Q = T\Gamma T\Gamma(\bigcap_{\alpha\in\Delta} Q_\alpha) = \bigcap_{\alpha\in\Delta} Q_\alpha\Gamma T\Gamma T \subseteq Q_\alpha\Gamma T\Gamma T$.

Then $T\Gamma T\Gamma Q \cap (T\Gamma Q\Gamma T \cup T\Gamma T\Gamma Q\Gamma T\Gamma T)\cap Q\Gamma T\Gamma T$
$\subseteq T\Gamma T\Gamma Q_\alpha \cap (T\Gamma Q_\alpha\Gamma T \cup T\Gamma T\Gamma Q_\alpha\Gamma T\Gamma T)\cap Q_\alpha\Gamma T\Gamma T \subseteq Q_\alpha$ for each $\alpha\in\Delta$.

Therefore $T\Gamma T\Gamma Q \cap (T\Gamma Q\Gamma T + T\Gamma T\Gamma Q\Gamma T\Gamma T \cap Q\Gamma T\Gamma T \subseteq \bigcap_{\alpha\in\Delta} Q_\alpha = Q$.

Therefore $Q = \bigcap_{\alpha\in\Delta} Q_\alpha$, is a quasi-ternary Γ-ideal of T.

***Theorem* 4.8**:**An additive subsemigroup Q of a ternary Γ-semiring T is a quasi-ternary Γ-ideal of T if Q is the intersection of a right ternary Γ-ideal, a lateral ternary Γ-ideal, and a left ternary Γ- ideal of T.**

Proof: Let R be a right ternary Γ-ideal, M be a lateral ternary Γ-ideal, and L be a left ternary Γ-ideal of T such that $Q=R\cap M\cap L$. Then, by Lemmas 4.3 and 4.7, we find that Q is a quasi-ternary Γ-ideal of T.

The converse of Theorem 4.8 does not hold, in general. But, in particular, we have the following result.

***Theorem* 4.9: AnadditivesubsemigroupQ ofaternaryΓ-semiring T isaminimalquasi-ternary Γ-ideal of T if and only if Q is the intersection of a minimal right ternary Γ-ideal, a minimal lateral ternary Γ-ideal, and a minimal left ternary Γ-ideal of T.**
Proof: Let R be a minimal right ternary Γ-ideal, M a minimal lateral ternary Γ-ideal, and L a minimal left ternary Γ-ideal of T such that $Q = R\cap M\cap L$.
Then, by Theorem 4.8, it follows that Q is a quasi-ternary Γ-ideal of T.
Now it remains to show that Q is minimal.
If possible, let $Q \subseteq Q$ be any other quasi-ternary Γ-ideal of T.
Then, $Q\Gamma T\Gamma T$ is a right ternary Γ-ideal of T and $Q\Gamma T\Gamma T \subseteq Q\Gamma T\Gamma T \subseteq R\Gamma T\Gamma T \subseteq R$.
Since R is a minimal right ternary Γ-ideal of T, we have $Q\Gamma T\Gamma T = R$.
Similarly, we can prove that $T\Gamma Q\Gamma T + T\Gamma T\Gamma Q\Gamma T\Gamma T = M$ and $T\Gamma T\Gamma Q = L$.
Therefore, $Q = R\cap M\cap L = Q\Gamma T\Gamma T\cap(T\Gamma Q\Gamma T+T\Gamma T\Gamma Q\Gamma T\Gamma T)\cap T\Gamma T\Gamma Q \subseteq Q$.
Consequently, Q= Q and hence Q is a minimal quasi-ternary Γ-ideal of T.
Conversely, let Q be a minimal quasi-ternary Γ-ideal of T.
Then, $Q\Gamma T\Gamma T\cap(T\Gamma Q\Gamma T+T\Gamma T\Gamma Q\Gamma T\Gamma T)\cap T\Gamma T\Gamma Q\subseteq Q$. Let $q \in Q$.
Then, $q\Gamma T\Gamma T$ is a right ternary Γ-ideal, $(T\Gamma q\Gamma T+T\Gamma Tq\Gamma T\Gamma T)$ is a lateral ternary Γ-ideal, and $T\Gamma T\Gamma q$ is a left ternary Γ-ideal of T.
Therefore, by Theorem 4.8, $q\Gamma T\Gamma T\cap(T\Gamma q\Gamma T+T\Gamma T\Gamma q\Gamma T\Gamma T)\cap T\Gamma T\Gamma q$ is a quasi-ternary Γ-ideal of T, and $q\Gamma T\Gamma T\cap(T\Gamma q\Gamma T+T\Gamma T\Gamma q\Gamma T\Gamma T)\cap T\Gamma T\Gamma q\subseteq Q\Gamma T\Gamma T\cap(T\Gamma Q\Gamma T+T\Gamma T\Gamma Q\Gamma T\Gamma T)\cap T\Gamma T\Gamma Q\subseteq Q$.
Since Q is a minimal quasi-ternary Γ-ideal of T, we have
$q\Gamma T\Gamma T\cap(T\Gamma q\Gamma T+T\Gamma T\Gamma q\Gamma T\Gamma T)\cap T\Gamma T\Gamma q = Q$.
Now it remains to show that $q\Gamma T\Gamma T$, $(T\Gamma q\Gamma T+T\Gamma T\Gamma q\Gamma T\Gamma T)$, and $T\Gamma T\Gamma q$ are, respectively, a minimal right, a minimal lateral, and a minimal left ternary Γ-ideal of T.
If possible, let R be any right ternary Γ-ideal of T such that $R \subseteq q\Gamma T\Gamma T$. Then $R\Gamma T\Gamma T \subseteq R \subseteq q\Gamma T\Gamma T$.
Now, $R\Gamma T\Gamma T\cap(T\Gamma q\Gamma T+T\Gamma T\Gamma q\Gamma T\Gamma T)\cap T\Gamma T\Gamma q \subseteq q\Gamma T\Gamma T\cap(T\Gamma q\Gamma T+T\Gamma T\Gamma q\Gamma T\Gamma T)\cap T\Gamma T\Gamma q = Q$.
Thus, by minimality of Q, we find that $Q = R\Gamma T\Gamma T\cap(T\Gamma q\Gamma T+T\Gamma T\Gamma q\Gamma T\Gamma T)\cap T\Gamma T\Gamma q$.
This implies that $Q \subseteq R\Gamma T\Gamma T$. Again, $q\Gamma T\Gamma T \subseteq Q\Gamma T\Gamma T \subseteq (R\Gamma T\Gamma T)\Gamma T\Gamma T \subseteq R\Gamma T\Gamma T$.
Thus, $q\Gamma T\Gamma T = R\Gamma T\Gamma T \subseteq R$ and hence $R = q\Gamma T\Gamma T$. Consequently, $q\Gamma T\Gamma T$ is a minimal right ternary Γ-ideal of T. Similarly, we can prove that $(T\Gamma q\Gamma T+T\Gamma T\Gamma q\Gamma T\Gamma T)$ is a minimal lateral ternary Γ-ideal and $T\Gamma T\Gamma q$ is a minimal left ternary Γ-ideal of T.

Theorem **4.10: Any minimal lateral ternary Γ-ideal of a ternary Γ-semiring T is a minimal ternary Γ-ideal of T.**

Proof: Let M be a minimal lateral ternaryΓ-ideal of T. We will show that M is a minimal ternary Γ-ideal of T. Let $m \in$ M. Then, TΓmΓT+TΓTΓmΓTΓT is a lateral ternary Γ-ideal of T and TΓmΓT+TΓTΓmΓTΓT⊆TΓMΓT+ TΓTΓMΓTΓT ⊆ M. Since M is minimal, we have M = TΓMΓT + TΓTΓMΓTΓT.
Now, MΓTΓT = (TΓMΓT + TΓTΓMΓTΓT)ΓTΓT= (TΓMΓT)ΓTΓT+ (TΓTΓMΓTΓT)ΓTΓT ⊆TΓMΓT + TΓTΓMΓTΓT ⊆ M and TΓTΓM =ΓTΓT (TΓTΓMΓTΓT)=ΓTΓT(TΓMΓT)+ ΓTΓT (TΓTΓMΓTΓT)⊆TΓMΓT + TΓTΓMΓTΓT ⊆ M. This implies that M is both right ternary Γ-ideal and left ternary Γ-ideal of T. Consequently, M is aternary Γ-ideal of T. Now it remains to show that M is a minimal ternary Γ-ideal of T. If possible, let M be a ternary Γ-ideal of T such that M⊆M. SinceM is a ternary Γ-ideal of T, it is a lateral ternary Γ-ideal of T.
By hypothesis, we have M=M. Consequently, M is a minimal ternary Γ-ideal of T.

Corollary **4.11. Any minimal quasi-ternary Γ-ideal of a ternary Γ-semiring T is contained in a minimal ternary Γ-ideal of T.**

Proof: Let Q be a minimal quasi-ternary Γ-ideal of T. Then, by theorem 4.9, Q=R∩M∩L, where R is a minimal right ternary Γ-ideal, M a minimal lateral ternary Γ-ideal, and L a minimal left ternary Γ-ideal of T. Clearly, Q⊆M. By theorem 4.10, it follows that M is a minimal ternary Γ-ideal of T.

Theorem **4.12:Let x be an idempotent element of a ternary Γ-semiring T, that is, $x\Gamma x\Gamma x=x$. IfR is a right ternary Γ-ideal, M a lateral ternary Γ-ideal, and L a left ternary Γ-ideal of T, then RΓxΓx, xΓxΓMΓxΓx, and xΓxΓLare quasi-ternary Γ-ideals of T.**

Proof: To show RΓxΓx, xΓxΓMΓxΓx, and$x$ΓxΓLare quasi-ternary Γ-ideals of T, it is sufficient to show that

$$R\Gamma x\Gamma x = R \cap (T\Gamma x\Gamma T+T\Gamma T\Gamma x\Gamma T\Gamma T) \cap T\Gamma T\Gamma x,$$
$$x\Gamma x\Gamma M\Gamma x\Gamma x = x\Gamma T\Gamma T\cap M\cap T\Gamma T\Gamma x,$$
$$x\Gamma x\Gamma L= x\Gamma T\Gamma T\cap (T\Gamma x\Gamma T+T\Gamma T\Gamma x\Gamma T\Gamma T)\cap L.$$

For the first case, clearly we see that RΓxΓx⊆R∩TΓTΓx. Let $a \in$ R∩TΓTΓx.

Then, $a \in$ R and $a \in$ TΓTΓx. Now, $a \in$ TΓTΓx implies that $a = \sum_{i=1}^{n} s_i\alpha_i t_i\beta_i x$ for some $s_i,t_i\in$ T and $\alpha_i, \beta_i \in \Gamma$.

Therefore, $a\alpha x\beta x = (\sum_{i=1}^{n} s_i\alpha_i t_i\beta_i x)x\beta x = \sum_{i=1}^{n} s_i\alpha_i t_i\beta_i (x\alpha x\beta x) = \sum_{i=1}^{n} s_i\alpha_i t_i\beta_i x = a.$
Thus, it follows that a∈RΓxΓx and hence RΓxΓx= R∩TΓTΓx.
Again, $a = a\alpha x\beta x$∈TΓxΓT and 0∈TΓTΓxΓTΓT. So we find that a ∈(TΓxΓT+TΓTΓxΓTΓT).
Thus, R∩TΓTΓx ⊆ (TΓxΓT+TΓTΓxΓTΓT). Consequently, RΓxΓx = R ∩ (TΓxΓT+TΓTΓxΓTΓT) ∩ TΓTΓx.
For the second case, We see that xΓxΓMΓxΓx⊆xΓTΓT∩M∩TΓTΓx.
Let $a \in x$ΓTΓT∩M∩TΓTΓx. Then, $a \in x$ΓTΓT, $a \in$ M, and $a \in$ TΓTΓx.

Now, $a \in x\Gamma T\Gamma T$ and $a \in T\Gamma T\Gamma x \Rightarrow a = \sum_{i=1}^{m} x\alpha_i s_i\beta_i t_i = \sum_{j=1}^{n} u_j\alpha_j v_j\beta_j x$ for some $s_i,t_i,u_j,v_j\in$ T

and $\alpha_i, \beta_i, \alpha_j, \beta_j \in \Gamma$. Therefore,

$$x\alpha x\beta a\gamma x\delta x = x\alpha x(\sum_{i=1}^{m} x\alpha_i s_i\beta_i t_i)\gamma x\delta x = (\sum_{i=1}^{m} (x\alpha x\beta x)\alpha_i s_i\beta_i t_i)\gamma x\delta x$$

$$= (\sum_{i=1}^{m} x\alpha_i s_i\beta_i t_i)\gamma x\delta x = (\sum_{j=1}^{n} u_j\alpha_j v_j\beta_j x)\gamma x\delta x = \sum_{j=1}^{n} u_j\alpha_j v_j\beta_j (x\gamma x\delta x) = \sum_{j=1}^{n} u_j\alpha_j v_j\beta_j x = a.$$

Consequently, $a \in x$ΓxΓMΓxΓx and hence xΓxΓMΓxΓx = xΓTΓT∩M∩TΓTΓx.
The third case can be proved in the same way as in the first case.

Definition **4.13 :**An element a of a ternary Γ-semiring. T is said to be ***regular*** if there exist x∈ T, α, ∈Γ such that $a\alpha x\beta a = a$.

Definition **4.14 :** A ternary Γ-semiring T is said to be ***regular ternary Γ-semiring*** provided every element is regular.

Theorem 4.15: The following conditions in a ternary Γ-semiring T are equivalent:

(i) T is regular;

(ii) For any right ternary Γ-ideal R, lateral ternaryΓ-ideal M and left ternaryΓ-ideal L of T,

$R\Gamma M\Gamma L = R\cap M\cap L$;

(iii) For $a, b, c \in T$, $<a>_r\Gamma<b>_m\Gamma<c>_l = <a>_r \cap <b>_m \cap <c>_l$;

(iv) For $a \in T$, $<a>_r\Gamma<a>_m\Gamma<a>_l = <a>_r \cap <a>_m \cap <a>_l$.

Proof: (i) =⇒ (ii). Suppose T is a regular ternary Γ-semiring.

Let R, M and L be a right ternary Γ-ideal, a lateral ternary Γ-ideal and a left ternary Γ-ideal of T respectively. Then clearly, $R\Gamma M\Gamma L \subseteq R\cap M \cap L$.Now for $a\in R\cap M \cap L$, we have $a = a\alpha x\beta a$ for some $x\in T$, $\alpha, \beta\in\Gamma$. This implies that $a = a\alpha x\beta a = (a\alpha x\beta a)(x\alpha a\beta x)\delta(a\alpha x\beta a)\in R\Gamma M\Gamma L$. Thus we have $R\cap M \cap L\subseteq R\Gamma M\Gamma L$. So we find that $R\Gamma M\Gamma L = R\cap M \cap L$.
Clearly, (ii) ⇒ (iii) and (iii) ⇒ (iv).
To complete the proof, it remains to show that (iv) ⇒ (i).
Let $a\in$ T. Clearly, $a\in<a>_r \cap <b>_m \cap <c>_l = <a>_r\Gamma<b>_m\Gamma<c>_l$.
Then we have,$a \in (a\Gamma T\Gamma T + na)\Gamma(T\Gamma a\Gamma T + T\Gamma T\Gamma a\Gamma T\Gamma T + na)\Gamma(T\Gamma T\Gamma a + na) \subseteq a\Gamma T\Gamma a$.
So we find that $a \in a\Gamma T\Gamma a$ and hence there exists an elements $x \in T$ such that $a = a\alpha x\beta a$, for all $\alpha, \beta \in\Gamma$. This implies that a is regular and hence T is regular.

***Theorem* 4.16. If,for every quasi-ternary Γ-ideal Q of T, QΓQΓQ = Q,then T is a regular ternary Γ-semiring.**

Proof: If R is a minimal right ternary Γ-ideal, M a minimal lateral ternary Γ-ideal, and L a minimal left ternary Γ-ideal of T, then, byTheorem 4.9, it follows that $R\cap M\cap L$ is a quasi-ternary Γ-ideal of T. Now, by hypothesis,

$R\cap M\cap L= [(R\cap M\cap L)\Gamma]^2(R\cap M\cap L) =(R\cap M\cap L)\Gamma(R\cap M\cap L)\Gamma(R\cap M\cap L)\subseteq R\Gamma M\Gamma L$.

Again, clearly $R\Gamma M\Gamma L\subseteq R\cap M\cap L$.So,$R\cap M\cap L=R\Gamma M\Gamma L$ and hence, by Theorem 4.15, T is a regular ternary Γ-semiring.

Definition* 4.17**: A ternary Γ-subsemiring B of a ternary Γ-semiring T is called a ***bi-ternary* Γ*-ideal of T if $B\Gamma T\Gamma B\Gamma T B\subseteq B$.

***Lemma* 4.18: Every quasi-ternary Γ-ideal of a ternary Γ-semiring T is a bi-ternary Γ-ideal of T.**

Proof. Let Q be a quasi-ternary Γ-ideal of T. Then we see that$Q\Gamma T\Gamma Q\Gamma T\Gamma Q\subseteq Q\Gamma(T\Gamma T\Gamma T)\Gamma T \subseteq Q\Gamma T\Gamma T$, $Q\Gamma T\Gamma Q\Gamma T\Gamma Q\subseteq T\Gamma(T\Gamma T\Gamma T)\Gamma Q \subseteq T\Gamma T\Gamma Q$, and$Q\Gamma T\Gamma Q\Gamma T\Gamma Q\subseteq T\Gamma T\Gamma Q\Gamma T\Gamma T$. Again $\{0\}\subseteq T\Gamma Q\Gamma T$. So ,$Q\Gamma T\Gamma Q\Gamma T\Gamma Q \subseteq T\Gamma Q\Gamma T+T\Gamma T\Gamma Q\Gamma T\Gamma T$. Consequently, it follows that $Q\Gamma T\Gamma Q\Gamma T\Gamma Q \subseteq Q\Gamma T\Gamma T\cap(T\Gamma Q\Gamma T+ T\Gamma T\Gamma Q\Gamma T\Gamma T)\cap T\Gamma T\Gamma Q\subseteq Q$ and hence Q is a bi-ternary Γ-ideal of T.

***Note* 4.19:** The converse of Lemma 4.15 does not hold, in general, that is, a bi-ternary Γ-ideal of a ternary Γ-semiring T may not be a quasi-ternary Γ-ideal of T.

***Remark* 4.20:** Since every left, right, and lateral ternary Γ-ideal of T is a quasi-ternary Γ-ideal of T, it follows that every left, right, and lateral ternary Γ-ideal of T is a bi-ternary Γ-ideal of T, but the converse is not true, in general.

***Theorem* 4.21: If B is a bi-ternary Γ-ideal of a ternary Γ-semiring T and S is a ternary Γ-subsemiring of T, then $B\cap S$ is a bi-ternary Γ-ideal of T.**

***Lemma* 4.22: If B is a bi-ternary Γ-ideal of a ternary Γ-semiring T and S_1, S_2 are two ternary Γ-subsemirings of T, then $B\Gamma S_1\Gamma S_2$, $S_1\Gamma B\Gamma S_2$, and $S_1\Gamma S_2\Gamma B$ are bi-ternary Γ-ideals of T**.

***Corollary* 4.23: If B_1, B_2, andB_3 are three bi-ternary Γ-ideals of a ternary Γ-semiring T, then $B_1\Gamma B_2\Gamma B_3$ is a bi-ternary Γ-ideal of T.**

***Corollary* 4.24: If Q_1, Q_2, andQ_3 are three quasi-ternary Γ-ideals of a ternary Γ-semiring T, then $Q_1\Gamma Q_2\Gamma Q_3$ is a bi-ternary Γ-ideal of T.**

In general, if B is a bi-ternary Γ-ideal of a ternary Γ-semiring T and C is a bi-ternary Γ-ideal of B, thenC is not a bi-ternary Γ-ideal of T. But, in particular, we have the following result.

***Theorem* 4.25. Let B be a bi-ternary Γ-ideal of a ternary Γ-semiring T, andC a bi-ternary Γ-ideal of B such that CΓCΓC =C. Then C is a bi-ternary Γ-ideal of T**.

***Proof*:** Since B is a bi-ternary Γ-ideal of T, BΓTΓBΓTΓB⊆B, and since C is a bi-ternary Γ-ideal of B, CΓBΓCΓBΓC⊆C.Therefore,

$$\begin{aligned} C\Gamma T\Gamma C\Gamma T\Gamma C &= (C\Gamma C\Gamma C)\Gamma T\Gamma C\Gamma T\Gamma(C\Gamma C\Gamma C) \\ &= C\Gamma C\Gamma(C\Gamma T\Gamma C\Gamma T\Gamma C)\Gamma C\Gamma C \\ &\subseteq C\Gamma C\Gamma(B\Gamma T\Gamma B\Gamma T\Gamma B)\Gamma C\Gamma C \subseteq C\Gamma C\Gamma B\Gamma C\Gamma C \\ &= C\Gamma C\Gamma B\Gamma C\Gamma(C\Gamma C\Gamma C) \subseteq C\Gamma(C\Gamma B\Gamma C\Gamma B\Gamma C)\Gamma C \subseteq C\Gamma C\Gamma C = C. \end{aligned}$$

Definition* 4.26 :**. An element a of a ternary Γ-semiring Tis said to be Γ-invertible***in Tif there exists an element b in T (called the ***ternary* Γ*-semiring-inverse*** *of a*) such that $a\Gamma b\Gamma t = b\Gamma a\Gamma t = t\Gamma a\Gamma b = t\Gamma b\Gamma a = t$ for all $t \in$T.

Definition* 4.27 :**. A ternary Γ-semiring (Γ-ring) T with $|S| \geq 2$ is said to be a ***ternarydivision* Γ*-semiring*(Γ*-ring, resp.) if every non-zero element of Tis Γ-invertible.

***Theorem* 4.28: A ternary Γ-semiringThas no nonzero proper bi-ternary Γ-ideals if T is a ternary division Γ-semiring.**

Proof: Let T be a ternary division Γ-semiring and B be a nonzero bi-ternary Γ-ideal of T.
Let $a(\neq 0) \in$ B. Then there exists $s(\neq 0)\in$T such that $a\alpha s\beta x = s\alpha a\beta x = x\alpha a\beta s = x\alpha s\beta a = x$ for all $x \in$ T, $\alpha, \beta \in \Gamma$. This implies that T =BΓTΓT = TΓTΓB.

Now,
$$\begin{aligned} T &= B\Gamma T\Gamma T = B\Gamma(T\Gamma T\Gamma B)\Gamma(T\Gamma T\Gamma B) \\ &= B\Gamma(B\Gamma T\Gamma T)\Gamma(T\Gamma B\Gamma T)\Gamma(T\Gamma T\Gamma B)\Gamma B \\ &\subseteq B\Gamma(B\Gamma T\Gamma B\Gamma T\Gamma B)\Gamma B \subseteq B\Gamma B\Gamma B \subseteq B. \end{aligned}$$

Consequently, B = T and hence T has no nonzero proper bi-ternary Γ-ideals.

The converse of Theorem 4.28 is not true, in general. However, in particular, we havethe following result.

***Theorem* 4.29:A ternary Γ-semiringT is a ternary division Γ-semiring if T is MC and has nononzero proper bi-ternary Γ-ideals.**

Proof: Let T be an MC ternary Γ-semiring and has no nonzero proper bi-ternary Γ-ideals.
Let a(≠ 0)∈T. Then, $a\Gamma T\Gamma x$ and $x\Gamma a\Gamma T$ are two bi-ternary Γ-ideals of T for anynonzero $x\in$T. Since T is MC, it is ZDF. So, $a\Gamma T\Gamma x \neq \{0\}$and $x\Gamma a\Gamma T \neq \{0\}$.
By hypothesis, we have $a\Gamma T\Gamma x = x\Gamma a\Gamma T =$ T and hence for x (≠0)∈T, there exist $b,c \in$T, $\alpha, \beta \in \Gamma$, such that $a\alpha b\beta x = x\alpha a\beta c = x$. Let y be any element of T.
Then there exist $d, e \in$T, $\gamma, \delta \in \Gamma$ such that $a\gamma d\delta x = x\gamma a\delta e = y$.
Thus, $a\alpha b\beta y = a\alpha b(x\gamma a\delta e) = (a\alpha b\beta x)\gamma a\delta e = x\gamma a\delta e = y$ for all $y \in$T, $\alpha, \beta, \gamma, \delta \in \Gamma$.
Now, $(y\alpha a\beta b)\gamma a\delta b = y\alpha(a\beta b\gamma a)\delta b = y\alpha a\delta b$.
Since T is MC, we find that $y\alpha a\beta b = y$ for all $y \in$ T, $\alpha, \beta \in \Gamma$.
Similarly, we can show that $b\alpha a\beta y = y\alpha b\beta a =$ y for all y ∈T, $\alpha, \beta \in \Gamma$.
Thus, we find that $a\alpha b\beta y = y\alpha a\beta b = b\alpha a\beta y = y\alpha b\beta a = y$ for all $y\in$T, $\alpha, \beta \in \Gamma$ and hence T is a ternary division Γ-semiring.

***Theorem* 4.30: Let X, Y, and Z be three ternary Γ-sub semirings of a ternary Γ-semiring T and B = XΓYΓZ. Then, B is a bi-ternary Γ-ideal if at least one of X, Y, Z is a right, a lateral, or a left ternary Γ-ideal of T.**

Proof: Let B = XΓYΓZ. SupposeX is a right ternary Γ-ideal of T.
Then we find that (XΓYΓZ)ΓTΓ(XΓYΓZ)ΓTΓ(XΓYΓZ)
= XΓ(TΓTΓT)Γ(TΓTΓT)ΓTΓTΓYΓZ⊆XΓ(TΓTΓT)ΓTΓYΓZ⊆(XΓTΓT)ΓYΓZ⊆XΓYΓZ.
Consequently, B=XΓYΓZ is a bi- ternary Γ-ideal of T. Now suppose that Y is a right ternary Γ-ideal of T.
Then (XΓYΓZ)ΓTΓ(XΓYΓZ)ΓTΓ(XΓYΓZ)⊆XΓYΓ(TΓTΓT)Γ(TΓTΓT)ΓTΓTΓZ ⊆ XΓYΓ(TΓTΓT)ΓTΓZ ⊆ XΓYΓTΓTΓZ ⊆ XΓYΓZ. This implies that B=XΓYΓZ is a bi-ternary Γ-ideal of T.
Again, if Z is a right ternary Γ-ideal of T, then
(XΓYΓZ)ΓTΓ(XΓYΓZ)ΓTΓ(XΓYΓZ)⊆(XΓYΓZ)Γ(TΓTΓT)Γ(TΓTΓT)ΓTΓT⊆(XΓYΓZ)Γ(TΓTΓT)ΓT
⊆XΓYΓ(ZΓTΓT)⊆XΓYΓZ. Consequently, B=XΓYΓZ is a bi-ternary Γ-ideal of T.
Similar proofs can be given for other cases.

***Corollary* 4.31: A ternary Γ-subsemiring B of T is a bi-ternary Γ-ideal of T if B = RΓMΓL, where R is aright ternary Γ-ideal, M is a lateral ternary Γ-ideal, and L is a left ternary Γ-ideal of T.**

***Theorem* 4.32: Let B be a ternary Γ-subsemiring of a ternary Γ-semiring T. If R is a right ternary Γ-ideal, M is a lateral ternary Γ-ideal, and L is a left ternary Γ-ideal of T such that $R\Gamma M\Gamma L \subseteq B \subseteq R\cap M\cap L$, then Bis a bi-ternary Γ-ideal of T.**

***Proof*:** $B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq (R\cap M\cap L)\Gamma T\Gamma(R\cap M\cap L)\Gamma T\Gamma(R\cap M\cap L)$
$\subseteq R\Gamma(T\Gamma M\Gamma T)\Gamma L \subseteq R\Gamma M\Gamma L \subseteq B$.

The following theorem gives a characterization of a regular ternary semiring *S* in termsof bi-ternary Γ-ideal and quasi-ternary Γ-ideal of *T*.

***Theorem* 4.33: The following conditions in a ternary Γ-semiring T are equivalent:**
(i) T is regular,
(ii) for every bi-ternary Γ-ideal B of T, $B\Gamma T\Gamma B\Gamma T\Gamma B = B$,

(iii) for every quasi-ternary Γ-ideal Q of T, $Q\Gamma T\Gamma Q\Gamma T\Gamma Q = Q$.

Proof:(i)⇒(ii). Suppose *T*is regular. Let *B* be a bi-ternary Γ-ideal of *T*. Let $b \in B$. Then thereexists $x \in$ T, such that $a = a\alpha x\beta a$ for all $\alpha, \beta \in \Gamma$. This implies that $a = a\alpha x\beta a\gamma x\delta a \in B\Gamma T\Gamma B\Gamma T\Gamma B$. So we find that$B \subseteq B\Gamma T\Gamma B\Gamma T\Gamma B$. Again, since *B* is a bi-ternary Γ-ideal of *T*, $B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B$. Consequently, $B\Gamma T\Gamma B\Gamma T\Gamma B= B$.
Clearly, (ii)⇒(iii), by using Lemma 4.18.
(iii)⇒(i). Suppose (iii) holds. Let *R* be a right ternary Γ-ideal,*M* a lateral ternary Γ-ideal, and *L* a left ternary Γ-ideal of T. Then, $Q = R\cap M\cap L$ is a quasi-ternary Γ-ideal of *T*, by Theorem 4.8. By hypothesis, $Q\Gamma T\Gamma Q\Gamma T Q = Q$.Now, $R\cap M\cap L = Q = Q\Gamma T\Gamma Q\Gamma T Q \subseteq R\Gamma T\Gamma M\Gamma T\Gamma L \subseteq R\Gamma M\Gamma L$. Again, clearly $R\Gamma M\Gamma L \subseteq R\cap M\cap L$.So,$R\cap M\cap L = R\Gamma M\Gamma L$, and hence, by Theorem 4.15, *T*is a regular ternary Γ-semiring.

***Theorem* 4.34:A ternary Γ-sub semiring B of a regular ternary Γ-semiringT is a bi-ternary Γ-ideal of T ifand only if $B = B\Gamma T\Gamma B$.**

***Proof*:**If $B = B\Gamma T\Gamma B$, then it is easy to see that *B* is a bi-ternary Γ-ideal of *T*.

Conversely, suppose that *B* is a bi-ternary Γ-ideal of a regular ternary Γ-semiring *T*. Let $b \in B$, thenthere exists $x \in$Tsuch that $b = b\alpha x\beta b$, for $\alpha, \beta \in \Gamma$. This implies that
$b \in B\Gamma T\Gamma B$ and hence $B \subseteq B\Gamma T\Gamma B$. Again,$B\Gamma T\Gamma B \subseteq B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B$. Thus we find that
$B = B\Gamma T\Gamma B$.

***Theorem* 4.35:A ternary Γ-sub semiring B of a regular ternary Γ-semiringT is a bi-ternary Γ-ideal of T ifand only if B is a quasi-ternary Γ-ideal of T.**

***Proof*:** Let T be a regular ternary Γ-semiring. If *B* is a quasi-ternary Γ-ideal of T, then, from Lemma4.18, it follows that *B* is a bi-ternary Γ-ideal of T.

Conversely, let *B* be a bi-ternary Γ-ideal of *T*. From Theorem 4.15, we find that if T is a regular ternary Γ-semiring, then $R\cap M\cap L = R\Gamma M\Gamma L$ for any right ternary Γ-ideal *R*, any lateral ternary Γ-ideal *M*, andany left ternary Γ-ideal *L*.
Now,

$$B\Gamma T\Gamma T\cap(T\Gamma B\Gamma T+T\Gamma T\Gamma B\Gamma T\Gamma T)\cap T\Gamma T\Gamma B$$
$$= B\Gamma T\Gamma T\Gamma(T\Gamma B\Gamma T + T\Gamma T\Gamma B\Gamma T\Gamma T)\Gamma T\Gamma T\Gamma B$$
$$= B\Gamma(T\Gamma T\Gamma T)\Gamma B\Gamma(T\Gamma T\Gamma T)\Gamma B + B\Gamma(T\Gamma T\Gamma T)\Gamma T\Gamma B\Gamma(T\Gamma T\Gamma T)\Gamma T\Gamma B$$
$$\subseteq B\Gamma T\Gamma B\Gamma T\Gamma B + B\Gamma T\Gamma T\Gamma B\Gamma T\Gamma T\Gamma B$$
$$\subseteq B + B\Gamma T\Gamma B \text{ (since } B \text{ is a bi-ternary } \Gamma\text{-ideal)} = B + B \text{ (by Theorem 4.34)}$$
$$\subseteq B.$$

Consequently, *B* is a quasi-ternary Γ-ideal of T.
In view of Lemma 4.22 and Theorem 4.35, we have the following result.

***Theorem* 4.36:If $Q_1$and $Q_2$are two ternary Γ-sub semiring and $Q_3$is a bi-ternary Γ-ideal of a regular ternary Γ-semiring T, then $Q_1\Gamma Q_2\Gamma Q_3$, $Q_1\Gamma Q_3\Gamma Q_2$, and $Q_3\Gamma Q_1\Gamma Q_2$ are quasi-ternary Γ-ideals of T.**

In view of Corollary 4.24 and Theorem 4.36, we have the following result.

***Corollary* 4.37:For any three quasi-ternary Γ-ideals Q_1, Q_2, $Q_3$of a regular ternary Γ-semiring T, $Q_1\Gamma Q_2\Gamma Q_3$ is a quasi-ternary Γ-ideal of T.**

Conclusion :

In this paper mainly we start the study of quasi-ternary Γ-ideals, bi-ternary Γ-ideals in ternary Γ-semirings. We characterize those ternary Γ-ideals.

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