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Factorization of Cosine Function

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Abstract
The notion of convergence in $H(G)$ can be used to solve the following problem. Let sequence $\{a_k\}$ be a sequencein G which has no limit point in G and $\{m_k\}$ be a sequence of integers. Is there a function f which is analytic on G and such that only zeros of f are at the points a_k with multiplicity of zero at a_K equal to m_k ? The answer to such type of problem is due to Weierstrass Factorization theorem

I. Introduction

The power of complex methods is exhibited by an efficient theorem of complex analysis, Weierstrass factorization theorem . In the first section, we introduce notation and representation of infinite product . The Weierstrass Factorization Thorem is stated in section 2 and in section 3 , Factorization of cosine function is displayed using Weierstrass Factorization Theorem.

II. The Infinite Product

Let z_n be a sequence of complex numbers and if $z = lim\Pi_{k=1}^{n} z_k$ exists, then z is called the infinite poduct of the nmbersz_n and it is denoted by

$$z = \Pi_{n=1}^{\infty} z_n$$

Definition 1. *An elementary factor is one of the following functions $E_p(z)$ for $p = 0, 1, 2, 3, \ldots,$*

$$E_0(z) = 1 - z$$
$$E_p(z) = (1 - z)exp(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}),\ p \geq 1.$$

Note that the function $E_p(z)$ has a simple zero at $z = 1$ and no other zero .

Lemma 1. *If $|z| \leq 1$ and $p \geq 0$ then $|1 - E_p(z)| \leq |z|^{p+1}$.*

Theorem 1. *Let $\{a_n\}$ be a sequence in C such that $lim|a_n| = \infty$ and $a_n \neq 0$ for all $n \geq 1$. If sequence$\{p_n\}$ is any sequence of integers such that*

$$\sum_{n=1}^{\infty} (\frac{r}{|a_n|})_n^p + 1 < \infty \ldots (1)$$

for all $r > 0$ then

$$f(z) = \Pi_{n=1}^{\infty} E_{p_n}(\frac{z}{a_n})$$

converges in $H(c)$. The function f is an entire function with zeros only at the points a_n. If z_0 occurs in the sequence a_n exactly m times then f has a zero at $z = z_0$ of multiplicity m. If $p_n = n - 1$ then (1) will be satisfied.

Theorem 2. *Let sequence $\{f_n\}$ is a sequence in $H(G)$ and f belongs to $C(G, C)$ such that $f_n \to f$ then f is analytic and $f_n^k \to f^k$ for each integer $k \geq 1$. Here $H(G)$ denote the set of all analytic functions on G and $C(G, C)$ denote set of all continuous functions from G to C.*

Lemma 2. *Suppose G is an open set and $\{f_n\}$ is a sequence in $H(G)$ such that $f(z) = \Pi_{n=1}^{\infty} f_n(z)$ converges in $H(G)$. Assume that f is not the identically zero function and let K be a compact subset of G such that $f(z) \neq 0$ for all $z \in K$, then*

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{f_n(z)}$$

and the convergence is uniform over K.

Proof:- Since the infinite product converges in $H(G)$ to $f(z)$

$$f(z) = lim \Pi_{k=1}^{n} f_k(z)$$

$$\text{then } \frac{f'(z)}{f(z)} = lim \sum_{k=1}^{n} \frac{f_k'(z)}{f(z)}$$

$$= \sum_{k=1}^{\infty} \frac{f_k'(z)}{f(z)}$$

and the convergence is uniform over K.

Theorem 3 (The Weierstrass Factorization Theorem). *Let f be an entire function and let sequence$\{a_n\}$ be the non-zero zeros of f repeated according to multiplicity; suppose f has a zero at $z = 0$ of order $m \geq 0$. Then there is an entire function g and a sequence of integers $\{p_n\}$ such that*

$$f(z) = z^m e^{g(z)} \Pi_{n=1}^{\infty} E_{p_n}(\frac{z}{a_n})$$

III. Factorization of the Cosine Function

In this section an application of the Weierstrass Factorization Theorem to $cos\pi z$ is given

Theorem 4. *$cos\pi z$ can be factored as*

$$cos\pi z = \Pi_{n=1}^{\infty}\left[1 - \frac{4z^2}{(2n-1)^2}\right]$$

and the convergence is uniform over compact subsets of C.

Proof: The zeros of $cos\pi z = \frac{1}{2}(e^{i\pi z} + e^{-i\pi z})$ are exactly at $z = \frac{(2n-1)}{2}, n \in z$, ; moreover each zero is simple . Since

$$\sum_{-\infty}^{\infty}\left(\frac{r}{\frac{2n-1}{2}}\right)^2 < \infty \text{ , for all } r > 0$$

. By theorem (1) one can choose the Weierstrass factorization theorem for all n and choose $p_n = 1$, then

$$cos\pi z = e^{g(z)}\Pi_{n=-\infty}^{\infty}E_1\left(\frac{z}{\frac{2n-1}{2}}\right)$$

$$= e^{g(z)}\Pi_{n=-\infty}^{\infty}\left(1 - \frac{z}{\frac{2n-1}{2}}\right)e^{\left(\frac{z}{\frac{2n-1}{2}}\right)}$$

The terms of the product can be rearranged

$$cos\pi z = e^{g(z)}\Pi_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right) \ldots (2)$$

for some entire function $g(z)$. If $f(z) = cos\pi z$, then according to lemma (2)

$$-\pi tan\pi z = \frac{f'(z)}{f(z)}$$

$$= g'(z) + 4\sum_{n=1}^{\infty}\frac{2z}{(4z^2-(2n-1)^2)}$$

and the convergence is uniform over compact subsets of the plane. But

$$-\pi tan\pi z = 4\sum_{n=1}^{\infty}\frac{2z}{(4z^2-(2n-1)^2)}$$

So it must be that gis a constant, say $g(z) = a$, for all z. It follows from (2) that for $0 < |z| < 1$

$$cos\pi z = e^a \Pi_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)$$

Letting z approach zero gives that $e^a = 1$. This gives that

$$cos\pi z = \Pi_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)$$

Reference

John B.Conway . Functions of one Complex Variable , Second Edition, Springer International Student Edition