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Some Generalization of Eneström-Kakeya Theorem

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Abstract: *In this paper we prove some extension of the Eneström-Kakeya theorem by relaxing the hypothesis of this result in several ways and obtain zero-free regions for polynomials with restricted coefficients and there by present some interesting generalizations and extensions of the Enestrom-Kakeya Theorem.*

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1. INTRODUCTION

The well known Results Eneström-Kakeya theorem [1,2] in theory of the distribution of zeros of polynomials is the following.

Theorem ($\boldsymbol{A_1}$).): Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n* such that $0 < a_0 \le a_1 \le a_2 \le, \dots, \le a_n$ then all the zeros of P(z) lie in |z|≤1.

Applying the above result to the polynomial $z^n P(\frac{1}{z})$ we get the following result:

Theorem (A_2). If P(z) =$\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0 < a_n \le a_{n-1} \le a_{n-2} \le, \dots, \le a_0$

then P(z) does not vanish in |z|<1

In the literature [3-9], there exist several extensions and generalizations of the Enestrom-Kakeya Theorem. Recently B. A. Zargar [9] proved the following results:

Theorem (A_3). If P(z) =$\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that for some k ≥1,

$0 < a_n \le a_{n-1} \le a_{n-2} \le, \dots, \le a_0$ then P(z) does not vanish in the disk |z|<$\frac{1}{2k-1}$.

Theorem (A_4). *If* P(z) = $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that for some real number*

0 ≤*ρ*<*an*, $0 < a_n - \rho \le a_{n-1} \le a_{n-2} \le, \dots, a_1 \le a_0$ *then P(z) does not vanish in the disk* |z|<$\frac{1}{1+\frac{2\rho}{a_0}}$.

Theorem (A_5). *If* P(z) = $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that for some real number* k ≥1,
$0 < a_0 \le a_1 \le a_2 \le, \dots, \le ka_n$ *then P(z) does not vanish in the disk* |z|<$\frac{1}{2ka_n-\rho}$

Theorem (A_6). *If* P(z) = $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that for some real number* ρ ≥0

$0 < a_0 \le a_1 \le a_2 \le, \dots, \le a_{n-1} \le a_n + \rho$ *then P(z) does not vanish in the disk* |z|<$\frac{1}{2(a_n+\rho)-a_o}$

In this paper we give generalizations of the above mentioned results. In fact, we prove the following results:

Theorem 1. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some k ≥1

$\rho \geq 0$, $a_m \neq 0$, $a_n - \rho \leq a_{n-1} \leq, \ldots, \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq, \ldots, \geq a_1 \geq a_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{2k(a_m+|a_m|)-(a_0+2|a_m|+a_n)+a_n+2\rho}$

Corollary 1.. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $a_n \leq a_{n-1} \leq, \ldots, \leq a_{m+1} \leq a_m \geq a_{m-1} \geq, \ldots, \geq a_1 \geq a_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{2a_m+|a_n|-(a_0+a_n)}$

Corollary 2. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with positive real coefficients such that for some k ≥1 , $\rho \geq 0$, $a_m \neq 0$, $a_n - \rho \leq a_{n-1} \leq, \ldots, \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq, \ldots, \geq a_1 \geq a_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{2(2k-1)a_m-a_0+2\rho}$

Remark 1.

(i) By taking $\rho = 0$ and k = 1 in theorem 1, then it reduces to Corollary 1.

(ii) By taking $a_i > 0 \; for \; i = 0,1,2, \ldots, n-1$, theorem 1, then it reduces to Corollary 2.

Theorem 2. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some

$\rho \geq 0$, $0 < r \leq 1$, $a_n \leq a_{n-1} \leq, \ldots, \leq a_{m+1} \leq a_m + \rho \geq a_{m-1} \geq, \ldots, \geq a_1 \geq ra_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{4\rho+|a_0|+2a_m+|a_n|-a_n-r(a_0+|a_0|)}$.

Corollary 3. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some

$0 < r \leq 1$, $a_n \leq a_{n-1} \leq, \ldots, \leq a_{m+1} \leq a_m + \rho \geq a_{m-1} \geq, \ldots, \geq a_1 \geq ra_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{|a_0|+2a_m+|a_n|-a_n-r(a_0+|a_0|)}$.

Corollary 4. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with positive real coefficients such that for some

$\rho \geq 0$, $0 < r \leq 1$, $a_n \leq a_{n-1} \leq, \ldots, \leq a_{m+1} \leq a_m + \rho \geq a_{m-1} \geq, \ldots, \geq a_1 \geq ra_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_o}{4\rho+a_o+2a_m-2ra_0}$.

Remark 2.

(i) By taking $\rho = 0$ in theorem 2 then it reduces to Corollary 3.
(ii) By taking $a_i > 0 \; for \; i = 0,1,2,\dots,n-1$, in theorem 2, then it reduces to Corollary 4.

Theorem 3. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $0 < r \leq 1$,

$\rho \geq 0$, $a_m \neq 0$, $a_n + \rho \geq a_{n-1} \geq, \dots, \geq a_{m+1} \geq r a_m \leq a_{m-1} \leq, \dots, \leq a_1 \leq a_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{a_0 + 2|a_m| - 2r(a_m + |a_m|) + a_n + |a_n| + 2\rho}$.

Corollary 5. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some , $a_n \geq a_{n-1} \geq, \dots, \geq a_{m+1} \geq a_m \leq a_{m-1} \leq, \dots, \leq a_1 \leq a_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{a_0 - 2a_m + a_n + |a_n|}$.

Corollary 6. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with Positive real coefficients such that for some

$0 < r \leq 1$, $\rho \geq 0$, $a_m \neq 0$, $a_n + \rho \geq a_{n-1} \geq, \dots, \geq a_{m+1} \geq r a_m \leq a_{m-1} \leq, \dots, \leq a_1 \leq a_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_o}{a_0 + 2(a_n + \rho + (1-2r)a_m)}$.

Remark 3.

(i) By taking $\rho = 0$ and r = 1 in theorem 3, then it reduces to Corollary 5.
(ii) By taking $a_i > 0 \; for \; i = 0,1,2,\dots,n-1$, in theorem 3, then it reduces to Corollary 6..

Theorem 4. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $k \geq 1$,

$\rho \geq 0$, $a_n \geq a_{n-1} \geq, \dots, \geq a_{m+1} \geq a_m - \rho \leq a_{m-1} \leq, \dots, \leq a_1 \leq k a_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{|a_n| + a_n + k(a_0 + |a_0|) - |a_0| - 2a_m + 4\rho}$.

Corollary 7. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $k \geq 1$,

$a_n \geq a_{n-1} \geq, \dots, \geq a_{m+1} \geq a_m - \rho \leq a_{m-1} \leq, \dots, \leq a_1 \leq k a_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{|a_n| + a_n + k(a_0 + |a_0|) - |a_0| - 2a_m}$.

Corollary 8. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with positive real coefficients such that for some

$k \geq 1$, $\rho \geq 0$, $a_n \geq a_{n-1} \geq, \dots, \geq a_{m+1} \geq a_m - \rho \leq a_{m-1} \leq, \dots, \leq a_1 \leq k a_0$

then all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{(2k-1)a_0-2a_m+2a_n+4\rho}$.

Remark 4.

(i) By taking $\rho = 0$ in theorem 4, then it reduces to Corollary 7.
(ii) By taking $a_i > 0\ for\ i = 0,1,2,\dots,n-1$, in theorem 4, then it reduces to Corollary 8.

2. Proofs of the Theorems

Proof of the Theorem 1.

Let P(z)= $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$

Let Consider the polynomial J(z)=z^nP($\frac{1}{z}$)

And R(z)= (z-1)J(z) so that

Then R(z)=(z-1)($a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1}z^{n-m+1} + a_m z^{n-m} + a_{m+1}z^{n-m-1} + \cdots + a_{n-1}z + a_n$)

= $a_0 z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} + \cdots + (a_{n-1} - a_n)z + a_n\}$
Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} <\ for\ i = 0,1,2,\dots,n-1$.

Now $|R(z)| \geq |a_0||z|^{n+1} - \{ \ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n| \}$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{ |a_0 - a_1| + + \frac{|a_1-a_2|}{|z|} + \cdots + \frac{|a_{m-1}-a_m|}{|z|^{m-1}} + \frac{|a_m-a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-1}-a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{m-1} - ka_m + ka_m - a_m| + |a_m - ka_m + ka_m + a_{m+1}| + \cdots + |a_{n-1} + \rho - a_n - \rho| + |a_n| \}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{(a_1 - a_0) + (a_2 - a_1) + \cdots + (ka_m - a_{m-1}) + (k-1)|a_m| + (k-1)|a_m| + (ka_m - a_{m+1}) \dots + (a_{n-1} + \rho - a_n) + \rho + |a_n| \}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{ 2k(a_m - |a_m|) - (a_0 + 2|a_m| + a_n) + |a_n| + 2\rho\}]$

> 0 if $|z| > \frac{1}{|a_0|}[\ (2k(a_m - |a_m|) - (a_0 + 2|a_m| + a_n) + |a_n| + 2\rho\ ]$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}[\ (2k(a_m - |a_m|) - (a_0 + 2|a_m| + a_n) + |a_n| + 2\rho\ ]$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.
Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}[\ (2k(a_m - |a_m|) - (a_0 + 2|a_m| + a_n) + |a_n| + 2\rho\ ]$$

Since P(z) = z^nJ($\frac{1}{z}$) it follows, by replacing z by $\frac{1}{z}$,

Then all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{(2k(a_m-|a_m|)-(a_0+2|a_m|+a_n)+|a_n|+2\rho}$$

Hence P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{(2k(a_m - |a_m|) - (a_0 + 2|a_m| + a_n) + |a_n| + 2\rho}$$

This completes the proof of the Theorem 1.

Proof of the Theorem 2.

Let P(z)= $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

Let Consider the polynomial J(z)=z^nP($\frac{1}{z}$)

And R(z)= (z-1)J(z) so that

Then R(z)=(z-1)($a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \cdots + a_{n-1} z + a_n$)

= $a_0 z^{n+1} - \{(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m+1} + (a_m - a_{m+1}) z^{n-m} + \cdots + (a_{n-1} - a_n) z + a_n\}$

Also if $|z| > 1$ then $\frac{\mathbf{1}}{|z|^{n-i}} < for\ i = 0,1,2, \ldots, n-1.$

Now $|R(z)| \geq |a_0||z|^{n+1} - \{ \ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n| \}$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{ |a_0 - a_1| + + \frac{|a_1 - a_2|}{|z|} + \cdots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{ | ra_0 - a_1 + ra_0 \ + a_0 \ | + |a_1 - a_2| + \cdots + |a_{m-1} - \rho + \rho - a_m| + |a_m - \rho + \rho +$ *am+1+… +|an−1−an|+| an|*}]

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{(a_1 - ra_0) + \ (1-r)|a_0| + (a_2 - a_1) + \cdots + (a_m + \rho - a_{m-1}) + \rho + (a_m + \rho - a_{m+1}) + \rho \ldots + (a_{n-1} - a_n) + | \ a_n| \}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|} \{ 4\rho + |a_0| + 2a_m + |a_n| - a_n \ - r(a_0 \ + |a_0|) \}]$
> 0 if $|z| > \frac{1}{|a_0|}[\ 4\rho + \ |a_0| \ + \ 2a_m \ + \ |a_n| - \ a_n \ - r(a_0 \ + |a_0|)]$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}[\ 4\rho + \ |a_0| \ + \ 2a_m \ + \ |a_n| - \ a_n \ - r(a_0 \ + |a_0|)]$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.
Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}[\ 4\rho + \ |a_0| \ + \ 2a_m \ + \ |a_n| - \ a_n \ - r(a_0 \ + |a_0|)]$$

Since P(z) = z^nJ($\frac{1}{z}$) it follows, by replacing z by $\frac{1}{z}$,
Then all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{4\rho + |a_0| + 2a_m + |a_n| - a_n - r(a_0 + |a_0|)}$$

Hence P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{4\rho + |a_0| + 2a_m + |a_n| - a_n - r(a_0 + |a_0|)}$$

This completes the proof of the Theorem 2.

Proof of the Theorem 3.

Let P(z)= $a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$

Let Consider the polynomial J(z)=z^nP($\frac{1}{z}$)

And R(z)= (z-1)J(z) so that

Then R(z)=(z-1)($a_0z^n + a_1z^{n-1} + \cdots + a_{m-1}z^{n-m+1} + a_mz^{n-m} + a_{m+1}z^{n-m-1} + \cdots + a_{n-1}z + a_n$)

= $a_0z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} + \cdots + (a_{n-1} - a_n)z + a_n\}$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < for\ i = 0,1,2, \dots, n-1.$
Now $|R(z)| \geq |a_0||z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n| \}$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{ |a_0 - a_1| + + \frac{|a_1-a_2|}{|z|} + \cdots + \frac{|a_{m-1}-a_m|}{|z|^{m-1}} + \frac{|a_m-a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-1}-a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{ |a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{m-1} - ra_m + ra_m - a_m| + |a_m - ra_m + ra_m +$ *am+1/+…+/an−1+ρ−an−ρ/+/ an/*$\}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{(a_0 - a_1) + (a_1 - a_2) + \cdots + (a_{m-1} - ra_m) + (1-r)|a_m| + (1-r)|a_m| + (a_{m+1} - ra_m) \dots + (a_n + \rho - a_{n-1}) + \rho + |a_n|\}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{ |a_0| + 2|a_m| - 2r(a_m + |a_m|) + |a_n| + a_n + 2\rho \}]$
> 0 if $|z| > \frac{1}{|a_0|}[|a_0| + 2|a_m| - 2r(a_m + |a_m|) + |a_n| + a_n + 2\rho]$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}[|a_0| + 2|a_m| - 2r(a_m + |a_m|) + |a_n| + a_n + 2\rho]$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.
Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}[|a_0| + 2|a_m| - 2r(a_m + |a_m|) + |a_n| + a_n + 2\rho]$$

Since P(z) = z^nJ($\frac{1}{z}$) it follows, by replacing z by $\frac{1}{z}$,
Then all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{|a_0| + 2|a_m| - 2r(a_m + |a_m|) + |a_n| + a_n + 2\rho}$$

Hence P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_0| + 2|a_m| - 2r(a_m + |a_m|) + |a_n| + a_n + 2\rho}$$

This completes the proof of the Theorem 3.

Proof of the Theorem 4.

Let P(z)= $a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$

Let Consider the polynomial J(z)=z^nP($\frac{1}{z}$)

And R(z)= (z-1)J(z) so that

Then R(z)=(z-1)($a_0z^n + a_1z^{n-1} + \cdots + a_{m-1}z^{n-m+1} + a_mz^{n-m} + a_{m+1}z^{n-m-1} + \cdots + a_{n-1}z + a_n$)

= $a_0z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} + \cdots (a_{n-1} - a_n)z + a_n\}$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < for\ i = 0,1,2, \dots, n-1.$

Now $|R(z)| \geq |a_0||z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n| \}$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{ |a_0 - a_1| + + \frac{|a_1-a_2|}{|z|} + \cdots + \frac{|a_{m-1}-a_m|}{|z|^{m-1}} + \frac{|a_m-a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-1}-a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{| ka_0 - a_1 + ka_0 + a_0 | + |a_1 - a_2| + \cdots + |a_{m-1} - \rho + \rho - a_m| + |a_m - \rho + \rho +$ *am+1+… +/an−1−an/+/ an/*}]

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{(ka_0 - a_1) + (k-1)|a_0| + (a_1 - a_2) + \cdots + (a_{m-1} + \rho - a_m) + \rho + (a_{m+1} + \rho - a_m) + \rho \dots + (a_n - a_{n-1}) + | a_n| \}]$

$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|a_n| + a_n + k(a_0 + |a_0|) - |a_0| - 2a_m + 4\rho \}]$

> 0 if $|z| > \frac{1}{|a_0|}[|a_n| + a_n + k(a_0 + |a_0|) - |a_0| - 2a_m + 4\rho]$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}[|a_n| + a_n + k(a_0 + |a_0|) - |a_0| - 2a_m + 4\rho]$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.
Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}[|a_n| + a_n + k(a_0 + |a_0|) - |a_0| - 2a_m + 4\rho]$$

Since P(z) = z^nJ($\frac{1}{z}$) it follows, by replacing z by $\frac{1}{z}$,
Then all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{|a_n| + a_n + k(a_0 + |a_0|) - |a_0| - 2a_m + 4\rho}$$

Hence P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| + a_n + k(a_0 + |a_0|) - |a_0| - 2a_m + 4\rho}$$

This completes the proof of the Theorem 4.

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