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On The Properties Shared By a Simple Semigroup with an Identity $\zeta(S)$ and Any Semigroup with an Identity S^1

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ABSTRACT: *R. H. Bruck's theorem [1] established the fact that any semigroup S can be embedded in a Simple Semigroup which posses an identity element $\zeta(S)$. In this paper, we discuss some of the properties which $\zeta(S)$ shares with any semigroup which posses an identity element S^1. Thus we establish the following results*

i. *Any regular (inverse) semigroup can be embedded in a Simple regular (Inverse) semigroup with an identity element*
ii. *There exist simple inverse (and hence, regular) semigroups with an identity element which have an arbitrary cardinal number of D – classes.*

These results are new extensions arising from [1].

KEYWORDS: *Green's Relations; L, R, D, H and J, Simple Semigroups, Regular Semigroups, Inverse Semigroups.*

I. DEFINITIONS AND PRELIMINARIES

The elements of a Semigroup S, are said to be L – (R -) equivalent if and only if they generate the same principal left (right) ideal of S. We write $H = L \cap R \; and \; D = L^\circ R = R^\circ L$. Thus L, R, D, H and J are equivalence relations on S, such that $H \subseteq L \subseteq D \; and \; H \subseteq R \subseteq D$. We denote for each $a \in S$, L-class, R-class, H-class, D-class of a by La, Ra, Ha and Da respectively.
For any $a, b \in S, aJb \; if \; SaS \cup Sa \cup aS \cup \{a\} = SbS \cup Sb \cup bS \cup \{b\}$. (See [2] and [7])

a. S is a Simple Semigroup ⟺ S consists of a single J-class.
b. S is left [right] simple ⟺ S consists of a single L-[D-] class.
c. S is a Regular Semigroup if for each $a \in S \Rightarrow a \in aSa$.
d. S is an Inverse Semigroup if for each $a \in S$ there exists a unique element $x \in S$ such that $xax = x \; and \; axa = a$. Thus an inverse semigroup is a regular semigroup in which each element has a regular conjugate.

Comments

Every semigroup consists of a collection of mutually disjoint D-classes. Each D-class can be broken down in the following way called the egg-box picture. Imagine the elements of a D-class, arranged in a rectangular pattern so that the rows correspond to R-classes and the columns to L-classes contained in D. Each cell of the egg-box correspond to an H-class. A typical D-class looks like:

H_{11}	H_{12}	H_{13}	H_{14}	
....	....	....	....	R_2
....	....	....	....	R_3
L_1	L_2	L_3	L_4	

Figure 1: A typical D-class

Then a typical semigroup can be broken down as follows:

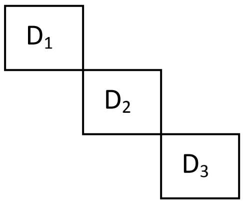

Figure 2: A typical semigroup

Remarks

a. S is simple [right simple] $\substack{\Rightarrow \\ \Leftarrow\neq}$ S is bisimple $\substack{\Rightarrow \\ \Leftarrow\neq}$ S is simple
b. The following conditions on a semigroup are equivalent
   i. S is regular and any two idempotents of S commute.
   ii. Every L-[R-] class of S contains a unique idempotent.
   iii. S is an inverse semigroup. (See [8] for proof)
c. For a semigroup S, we write; $S = \begin{cases} s, if\ s\ has\ an\ identity\ element \\ s \cup 1, otherwise \end{cases}$
d. Every semigroup consists of a collection of mutually disjoint D-classes. In a D-class each H-class is equally full of elements. Any two H-classes in the same D-class, have the same cardinal number.
e. In an inverse semigroup of idempotents, each D-class consists of a single idempotent (See [6])

S^1 can be embedded in $\zeta(S)$

Proof: Let $\zeta(S)$ be the semigroup generated by $S \cup \{a, b\}$where $a, b \in S$, such that $ab = 1, as = a, sb = s$ for every $s \in S^1$. Let $a^0 = 1, b^0 = 1$. Then the element of $\zeta(S)$ are of the form $b^i s a^j (s \in S, i\ and\ j\ are\ nonnegative\ numbers)$.

Hence $b^i s a^j = b^m t a^n \Leftrightarrow i = m, s = t\ and\ j = n$. Now, let $\alpha = b^i s a^j, \beta = b^m t a^n$, be any two elements of $\zeta(S)$, then $\alpha = b^i s a^{m-1}, \beta = b^{n-1} t a^j$.

Thus, $\zeta(S)$ is simple. Also 1 is an identity for$\zeta(S)$. Hence S^1 can be embedded in $\zeta(S)$. (See also, [1])

The L-, R- and D- classes of $\zeta(S)$ in terms of those of S^1

Let A and B be subsemigroups of $\zeta(S)$ such that $\begin{matrix} A = \{a^i, i = 0, 1, 2, 3, \dots\} \\ B = \{b^i, i = 0, 1, 2, 3, \dots\} \end{matrix}$

Then

Conjecture1: (See also, [3] and [4])

If $\{L_\lambda : \lambda \in \Lambda\}$ are the L-classes of S^1, then $\{BL_\lambda a^n : \lambda \in \Lambda, \mathrm{n} = 0, 1, 2, 3, \dots\}$ are the L-classes of $\zeta(S)$.

Proof: The elements $b^i s a^j$ and $b^m t a^n$ are L-equivalent in $\zeta(S) \Leftrightarrow$ there exists $b^p x a^q$ and $b^u v a^v$ in $\zeta(S)$ such that:

a. $b^p x a^q b^i s a^j = b^m t a^n$
b. $b^u v a^v b^m t a^n = b^i s a^j$

Thus, we have the following possibilities:

$$b^p x a^q b^i s a^j = \begin{cases} b^p x a^{j+q-1}, if\ q > i \\ b^p x s a^j, if\ q = i \\ b^{p+1-q} s a^j, if\ q < i \end{cases}$$

and

$$b^u y a^v b^m t a^n = \begin{cases} b^u y a^{n+v-m}, if\ v > m \\ b^u y t a^n, if\ v = m \\ b^{u+m-v} t a^n, if\ v < m \end{cases}$$

Suppose that $q > i$. Then from (a) we have that $j + (q - i) = n$ and (b) $n \leq j$. This is impossible! Hence $q \leq i$, and similarly $v \leq m$. Furthermore, each of them implies that $j = n$.

Since $q \leq i$, from (a), we have either $p = m\ and\ xs = t\ or\ p + (i - q) = m\ and\ s = t$. Since $v \leq m$, from (b) we either $u = i\ and\ yt = s\ or\ u + (m - v) = i\ and\ t = s$. For any non-negative integers i, m, we can find non-negative integers p, q, u, v satisfying these conditions. Hence we have shown that $b^i s a^j$ and $b^m t a^n$ are L-equivalent in $\zeta(S)$ if and only if $n = j$ and $sLt\ in\ S^1$.

Conjecture 2

If$\{Ri: i \in I\}$ are the R-classes of S^1 then $\{b^m RiA: i \in I, m = 0, 1, 2, 3, ...\}$ are the R-classes of $\zeta(S)$.

Proof: This is the left – right dual of Conjecture1.

Conjecture 3

If $\{D_\delta: \delta \in \Delta\}$ are the D-classes of S^1 then $\{BD_\delta A: \delta \in \Delta\}$ are the D-classes of $\zeta(S)$.

Proof:

The elements $b^i s a^j$ and $b^m t a^n$ are D-equivalent in $\zeta(S)$ if and only if there exists $b^p x a^q$ such that $b^i s a^j\ L\ b^p x a^q\ R\ b^m t a^n$. By Conjectures 1 and 2 above, this obtains if and only if $j = q, p = m$ and $sLxRt$ in S^1. Hence $b^i s a^j\ D\ b^m t a^n$in $\zeta(S)$ if and only if sDt in S^1.

Theorem

$\zeta(S)$ is a regular [inverse] semigroup if and only if S^1 is a regular [inverse] semigroup.[9]

Proof:
Let $b^i s a^j$ and $b^m t a^n$ be any two elements in $\zeta(S)$ with $s, t \in S^1$. Then we assert that

$$(b^i s a^j)(b^m t a^n)(b^i s a^j) = \begin{cases} b^i s^2 a^j, if\ j > m, n + (j - m) = i \\ b^i s t s a^j, if\ j = m, n = i \end{cases}$$

We also assert that these are the only cases for which the product on the left is equal to $b^i x a^j$ for any $x \in S^1$. Thus the inverse of $b^i s a^j$ in $\zeta(S)$ are the elements $b^j t a^i$ where t is an inverse of s in S^1. So $b^i s a^j$ has a unique inverse in $\zeta(S)$ if and only if s has a unique inverse in S^1. Hence the theorem is proved.

Extensions/Conclusion

a. Any regular [inverse] semigroup can be embedded in a simple regular [inverse] semigroup with identity.

Proof:
In view of the above theorem and the fact S^1 is a regular [inverse] semigroup if and only if S is a regular [inverse] semigroup, this extension is tenable.

b. There exist simple inverse [and hence, regular] semigroups with an identity, which contain an arbitrary number of D-classes.

Proof:
In view of the theorem above as well as Conjecture 3, it suffices to observe that in an inverse semigroup of idempotents, each D-class consists of a single idempotent. We also refer to Green's Theorem [7].

Green's Theorem: Let a and c be the D-equivalent elements of a semigroup S. Then there exists $\in S$ such that aRb and bLc and hence $as = b, bs^i = a, tb = c$, for some $s, s^i, t, t^i \in S^1$.

The functions $f: Ha \rightarrow Hc$ and $g: Hc \rightarrow Ha$ defined by $f(x) = txs$ and $g(y) = t^1ys^1$ are 1-1, onto, and mutually inverse. Hence, any two H-classes in the same D-class have the same cardinal number (See [5] and [10]). Thus this extension is tenable. ■

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