

Analytic Solutions of an Iterative Functional Differential Equation with Delays Depending On State

Jing Liu

Department of Mathematics, Binzhou University, Shandong 256603, P. R. China

ABSTRACT : This This paper is concerned with an iterative functional differential equation with the form

$$x'(z) = \frac{1}{x\left(az + \frac{b}{x(z)}\right)}, z \in C. \text{ By constructing a convergent power series solution of an auxiliary equation}$$

$$b^2[ag'(z) - \alpha g'(\alpha z)] = [g(\alpha^2 z) - ag(\alpha z)][g(\alpha z) - ag(z)]^2 g'(z), z \in C$$

the analytic solutions for the original equation are obtained. We not only discuss the constant given in Schröder transformation at resonance (i.e., at a root of the unity), but also discuss those near resonance (i.e., near a root of the unity) under Brjuno condition.

KEYWORDS: Iterative; functional differential equation; analytic solution; auxiliary equation; Brjuno condition.

2000 Mathematics Subject Classification: 39B12, 34A25, 30D05

I. INTRODUCTION

Delay differential equations with the form

$$x'(z) = f(z, x(z - \sigma(z))), \tag{1.1}$$

have been lucubrated in [1,2]. However, such equations have been relatively little researched, which delay function $\sigma(z)$ not only depends on the argument of unknown function, but also the state $\sigma(z) = \sigma(z, x(z))$. In this paper, we consider the equation

$$x'(z) = \frac{1}{x\left(az + \frac{b}{x(z)}\right)}, z \in C \tag{1.2}$$

in the complex field, where a and b are complex numbers, $x(z)$ denotes the unknown complex function. If set $f(z, x) = \frac{1}{x}$ and $\sigma(z) = (1-a)z - \frac{b}{x(z)}$, Eq (1.2) is the special case of (1.1). The purpose of this paper is to discuss the existence of analytic solutions of (1.2) in the complex field.

Let $a \neq 0, b = 0$, Eq.(1.1) changes to the functional differential equation

$$x'(z) = \frac{1}{x(az)}, \tag{1.3}$$

It is easy to know that Eq.(1.3) has a special solution $x(z) = \sqrt{2}a^{-\frac{1}{4}}z^{\frac{1}{2}}$.

A distinctive feature of Eq.(1.1) when $a \neq 0, b \neq 0$, is that the argument of the unknown function is dependent on the state, and this is the case we will emphasize on in this paper.

Let

$$y(z) = az + \frac{b}{x(z)}, \tag{1.4}$$

then

$$x(z) = \frac{b}{y(z) - az}.$$

Therefore, in view of Eq.(1.4), we have $x'(z) = \frac{b(a - y'(z))}{[y(z) - az]^2} \cdot \frac{1}{x(y(z))} = \frac{y(y(z)) - ay(z)}{b}$, and we admit

$$\frac{b(a - y'(z))}{[y(z) - az]^2} = \frac{y(y(z)) - ay(z)}{b}.$$

which is

$$b^2(a - y'(z)) = [y(z) - az]^2[y(y(z)) - ay(z)]. \tag{1.5}$$

To find analytic solution of (1.4), as in our previous works [3-13], our strategy remains to reduce Eq.(1.5) with $y(z) = g(\alpha g^{-1}(z))$, called the Schröder transformation[13] sometimes to the auxiliary equation

$$b^2[ag'(z) - \alpha g'(\alpha z)] = [g(\alpha^2 z) - ag(\alpha z)][g(\alpha z) - ag(z)]^2 g'(z), \tag{1.6}$$

a functional differential equation with proportional delay, and discuss analytic solution of Eq.(1.6) with the initial value condition

$$g(0) = \mu,$$

where μ is a complex number.

We will assume that α in Eq.(1.6) satisfies one of the following conditions:

$$(H_1) 0 < |\alpha| < 1;$$

$$(H_2) \alpha = e^{2\pi i \theta}, \text{ where } \theta \in \mathbb{R} \setminus \mathbb{Q} \text{ is a Brjuno number [14,15], i.e.}$$

$$B(\theta) = \sum_{k=0}^{\infty} \frac{\log q_{k+1}}{q_k} < +\infty \text{ where } \{p_k/q_k\} \text{ denotes the sequence of partial fraction of the continued}$$

fraction expansion of θ , said to satisfy the Brjuno condition;

$$(H_3) \alpha = e^{2\pi i q/p}, \text{ for some integer } p \in \mathbb{N}, \text{ with } p \geq 2 \text{ and } q \in \mathbb{Z} \setminus \{0\}, \text{ and } \alpha \neq e^{2\pi i l/k}, \text{ for all } 1 \leq k \leq p-1, \text{ and } l \in \mathbb{Z} \setminus \{0\}.$$

Observe that α is inside the unit circle S^1 in the case of (H_1) but on S^1 in the rest cases. More difficulties are encountered for α on S^1 since the small divisor $\alpha^n - 1$ is involved in the later (2.9). Under Diophantine condition: " $\alpha = e^{2\pi i \theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and there exist constants $\zeta > 0$ and $\sigma > 0$ such that $|\alpha^n - 1| \geq \zeta^{-1} n^{-\sigma}$ for all $n \geq 1$ ", The number $\alpha \in S^1$ is "far" from all roots of the unity, and was considered in different settings[11]. Since then, we have been striving to give a result in terms of analytic solutions for those α "near" a root of the unity, i.e., neither being roots of unity nor satisfying the Diophantine condition. The Brjuno condition in (H_2) provides such a chance for us, Moreover, we also discuss the so-called the resonance case. i.e..the case of (H_3) .

II. ANALYTIC SOLUTION OF THE AUXILIARY EQUATION

Theorem 2.1 Suppose (H_1) holds and $a \neq 0, 1, b \neq 0$. Then for any $\eta \in \mathbb{C} \setminus \{0\}$, Eq.(1.6) has an analytic solution with the form

$$g(z) = \mu + \eta z + \sum_{n=2}^{\infty} b_n z^n, \tag{2.1}$$

in a neighborhood of the origin, where $\mu = \sqrt[3]{\frac{b^2(a - \alpha)}{(1 - a)^3}}$.

Proof We seek a solution of Eq.(1.6) in a power series of the form

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{2.2}$$

where $b_0 = \mu$. By substituting (2.2) into (1.6), we get

$$\begin{aligned}
 & b^2 \sum_{n=0}^{\infty} (n+1)(a - \alpha^{n+1})b_{n+1}z^n \\
 &= \sum_{n=0}^{\infty} \left[\sum_{i=0}^n \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} (i+1)\alpha^j (\alpha^j - a)(\alpha^k - a)(\alpha^{n-i-j-k} - a)b_{i+1}b_jb_kb_{n-i-j-k} \right] z^n.
 \end{aligned}$$

Comparing coefficients we obtain

$$\begin{aligned}
 & b^2(n+1)(a - \alpha^{n+1})b_{n+1} \\
 &= \sum_{i=0}^n \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} (i+1)\alpha^j (\alpha^j - a)(\alpha^k - a)(\alpha^{n-i-j-k} - a)b_{i+1}b_jb_kb_{n-i-j-k}, n = 1, 2, \dots,
 \end{aligned}$$

that is

$$[b^2(a - \alpha) - (1 - a)^3 \mu^3]b_1 = 0, \tag{2.3}$$

and

$$\begin{aligned}
 & ab^2(n+1)(1 - \alpha^n)b_{n+1} \\
 &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} (i+1)\alpha^j (\alpha^j - a)(\alpha^k - a)(\alpha^{n-i-j-k} - a)b_{i+1}b_jb_kb_{n-i-j-k}, \\
 & n = 1, 2, \dots,
 \end{aligned} \tag{2.4}$$

In view of the definition of μ , we see that $b^2(a - \alpha) - (1 - a)^3 \mu^3 = 0$. So, we can choose b_1 to be η in (2.3). Once b_0 and b_1 are determined, the other terms of the sequence $\{b_n\}_{n=0}^{\infty}$ can be determined successively from (2.4) in a unique manner. Now, we show that the power series (2.1) converges in a neighborhood of the origin.

Since $0 < |\alpha| < 1$, we have

$$\left| \frac{(i+1)\alpha^j (\alpha^j - a)(\alpha^k - a)(\alpha^{n-i-j-k} - a)}{ab^2(n+1)(1 - \alpha^n)} \right| \leq \frac{(1+|a|)^3}{b^2|1 - \alpha^n|} \leq M$$

for some positive M . Thus, from (2.4) we obtain

$$|b_{n+1}| \leq M \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} |b_{i+1}| |b_j| |b_k| |b_{n-i-j-k}|, n = 1, 2, \dots, \tag{2.5}$$

If we define a sequence $\{B_n\}_{n=0}^{\infty}$, by $B_0 = |\mu|, B_1 = |\eta|$, and

$$|B_{n+1}| = M \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} B_{i+1} B_j B_k B_{n-i-j-k}, n = 1, 2, \dots,$$

then in view of (2.5), by induction, we can prove $|b_n| \leq B_n, n = 1, 2, \dots$. Now, we define the function

$$H(z, \mu, \eta, M) = \sum_{n=0}^{\infty} B_n z^n, \tag{2.6}$$

then

$$\begin{aligned}
 H^2(z, \mu, \eta, M) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n B_k B_{n-k} \right) z^n, \\
 H^3(z, \mu, \eta, M) &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \sum_{k=0}^{n-j} B_j B_k B_{n-j-k} \right) z^n,
 \end{aligned}$$

$$\begin{aligned}
 & H^4(z, \mu, \eta, M) \\
 &= (|\mu| + \sum_{n=0}^{\infty} B_{n+1} z^{n+1}) [\sum_{n=0}^{\infty} (\sum_{j=0}^n \sum_{k=0}^{n-j} B_j B_k B_{n-j-k}) z^n] \\
 &= |\mu| \sum_{n=0}^{\infty} (\sum_{j=0}^n \sum_{k=0}^{n-j} B_j B_k B_{n-j-k}) z^n + \sum_{n=0}^{\infty} (\sum_{i=0}^n \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} B_{i+1} B_j B_k B_{n-i-j-k}) z^{n+1} \\
 &= |\mu| H^3(z, \mu, \eta, M) + |\eta| |\mu|^3 z + \sum_{n=1}^{\infty} (|\mu|^3 B_{n+1} + \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} B_{i+1} B_j B_k B_{n-i-j-k}) z^{n+1} \\
 &= |\mu| H^3(z, \mu, \eta, M) + |\eta| |\mu|^3 z + |\mu|^3 \sum_{n=1}^{\infty} B_{n+1} z^{n+1} + \frac{1}{M} \sum_{n=1}^{\infty} B_{n+1} z^{n+1} \\
 &= |\mu| H^3(z, \mu, \eta, M) + |\eta| |\mu|^3 z + (|\mu|^3 + \frac{1}{M}) [H(z, \mu, \eta, M) - |\mu| - |\eta| z] \\
 &= |\mu| H^3(z, \mu, \eta, M) + (|\mu|^3 + \frac{1}{M}) H(z, \mu, \eta, M) - \frac{|\eta|}{M} z - |\mu| (|\mu|^3 + \frac{1}{M}),
 \end{aligned}$$

that is,

$$H^4(z, \mu, \eta, M) - |\mu| H^3(z, \mu, \eta, M) - (|\mu|^3 + \frac{1}{M}) H(z, \mu, \eta, M) + \frac{|\eta|}{M} z + |\mu| (|\mu|^3 + \frac{1}{M}) = 0, \tag{2.7}$$

Define the function

$$R(z, \zeta, \mu, \eta, M) = \zeta^4 - |\mu| \zeta^3 - (|\mu|^3 + \frac{1}{M}) \zeta + \frac{|\eta|}{M} z + |\mu| (|\mu|^3 + \frac{1}{M}). \tag{2.8}$$

For (z, ζ) from a neighborhood of $(0, |\mu|)$. Since $R(0, |\mu|, \mu, \eta, M) = 0$, and

$R'_\zeta(0, |\mu|, \mu, \eta, M) = \frac{1}{M}$, according to the implicit function theorem, there exists a unique function

$\zeta(z, \mu, \eta, M)$, analytic in a neighborhood of zero, such that

$$\zeta(0, \mu, \eta, M) = |\mu|, \quad \zeta'_\zeta(0, \mu, \eta, M) = |\eta|$$

and $R(z, \zeta, \mu, \eta, M) = 0$. By (2.6) and (2.7), we have $H(z, \mu, \eta, M) = \zeta(z, \mu, \eta, M)$. It follows that the power series (2.6), and hence also (2.1), converges in a neighborhood of the origin. The proof is complete.

Next we devote to the existence of analytic solution of (1.6) under the Brjuno condition. To do this, we first recall briefly the definition of Brjuno numbers and some basic facts. As stated in [15], for a real number θ , we let $[\theta]$ denote its integer part and $\{\theta\} = \theta - [\theta]$ denote its fractional part. Then every irrational number $[\theta]$ has a unique expression of the *Gauss'*, continued fraction

$$\theta = a_0 + \theta_0 = a_0 + \frac{1}{a_1 + \theta_1} = \dots,$$

denoted simply by $\theta = [a_0, a_1, \dots, a_n, \dots]$, where a_j 's and θ_j 's are calculated by the algorithm:

(a) $a_0 = [\theta], \theta_0 = \{\theta\}$, and (b) $a_n = [\frac{1}{\theta_{n-1}}], \theta_n = \{\frac{1}{\theta_{n-1}}\}$ for all $n \geq 1$. Define the sequences $(p_n)_{n \in \mathbb{N}}$ and

$(q_n)_{n \in \mathbb{N}}$ as follows:

$$\begin{aligned}
 q_{-2} &= 1, & q_{-1} &= 0, & q_n &= a_n q_{n-1} + q_{n-2}, \\
 p_{-2} &= 0, & p_{-1} &= 1, & p_n &= a_n p_{n-1} + p_{n-2}.
 \end{aligned}$$

It is easy to show that $p_n/q_n = [a_0, a_1, \dots, a_n, \dots]$. Thus, for every $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we associate, using its

convergence, an arithmetical function $B(\theta) = \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n}$. We say that θ is a Brjuno number or that it

satisfies Brjuno condition if $B(\theta) < +\infty$. The Brjuno condition is weaker than the Diophantine condition. For example, if $a_{n+1} \leq ce^{a_n}$ for all $n \geq 0$, where $c > 0$ is a constant, then $\theta = [a_0, a_1, \dots, a_n, \dots]$ is a Brjuno number but is not a Diophantine condition. So, the case (H_2) contains both Diophantine condition and a part of α "near" resonance. Let $\theta \in R \setminus Q$ and $(q_n)_{n \in N}$ be the sequence of partial denominators of the Gauss', continued fraction for θ . As in [15], let

$$A_k = \left\{ n \geq 0 \mid \|n\theta\| \leq \frac{1}{8q_k} \right\}, \quad E_k = \max(q_k, \frac{q_{k+1}}{4}), \quad \eta_k = \frac{q_k}{E_k}.$$

Let A_k^* be the set of integers $j \geq 0$ such that either $j \in A_k$ or for some j_1 and j_2 in A_k , with $j_1 - j_2 < E_k$, One has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any integer $n \geq 0$, define

$$l_k(n) = \max\left(\frac{(1 + \eta_k)n}{q_k} - 2, \frac{(m_n \eta_k + n)}{q_k} - 1\right),$$

where $m_n = \max\{j \mid 0 \leq j \leq n, j \in A_k^*\}$. We then define function $h_k : N \rightarrow R_+$ as follows:

$$h_k(n) = \begin{cases} \frac{m_n + \eta_k n}{q_k} - 1, & \text{if } m_n + q_k \in A_k^*, \\ l_k(n), & \text{if } m_n + q_k \notin A_k^*. \end{cases}$$

Let $g_k(n) := \max(h_k(n), [\frac{n}{q_k}])$, and define $k(n)$ by the condition $q_{k(n)} \leq n \leq q_{k(n)+1}$. Clearly, $k(n)$ is non-decreasing. Then we are able to state the following result.

Lemma 2.1 (Davies lemma [16]) Let $K(n) = n \log 2 + \sum_{k=0}^{k(n)} g_k(n) \log(2q_{k+1})$. Then

(a) there is a universal constant $\gamma > 0$ (independent of n and θ), such that

$$K(n) \leq n \left(\sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} + \gamma \right);$$

(b) $K(n_1) + K(n_2) \leq K(n_1 + n_2)$ for all n_1 and n_2 , and

(c) $-\log|\alpha^n - 1| \leq K(n) - K(n-1)$.

Theorem 2.2 Suppose (H_2) holds and $a \neq 0, 1, b \neq 0$. Then for any $\eta \in C \setminus \{0\}$, Eq.(1.6) has an analytic solution of the form (2.1) in a neighborhood of the origin, where μ is the same number defined in theorem 2.1.

Proof As in the proof of Theorem 2.1, we seek a power series solution of the form (2.1). Set $b_0 = \mu$ and $b_1 = \eta$. Then (2.4) again holds. From (2.4), we have

$$|b_{n+1}| \leq \frac{L}{|1 - \alpha^n|} \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} |b_{i+1}| |b_j| |b_k| |b_{n-i-j-k}|, \quad n = 1, 2, \dots, \tag{2.9}$$

where $L = \frac{(1 + |a|)^3}{b^2}$. To construct a majorant series of (2.1), we consider the implicit functional equation

$$R(z, \varphi, \mu, \eta, L) = 0, \tag{2.10}$$

where R is defined in (2.8). Similarly to the proof of Theorem 2.1, using the implicit function theorem we can prove that (2.10) has a unique analytic solution $\varphi(z, \mu, \eta, L)$ in a neighborhood of the origin such that $\varphi(0, \mu, \eta, L) = |\mu|$, and $\varphi'_z(0, \mu, \eta, L) = |\eta|$. Thus $\varphi(z, \mu, \eta, L)$ in (2.10) can be expanded into a convergent series

$$\varphi(z, \mu, \eta, L) = \sum_{n=0}^{\infty} C_n z^n, \tag{2.11}$$

in a neighborhood of the origin. Replacing (2.11) into (2.10) and comparing coefficients we obtain that $C_0 = |\mu|$, $C_1 = |\eta|$ and

$$c_{n+1} = L \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} C_{i+1} C_j C_k C_{n-i-j-k}, \quad n = 1, 2, \dots, \tag{2.12}$$

Note that the series (2.11) converges in a neighborhood of the origin. Hence there is a constant $T > 0$ such that

$$C_n \leq T^n, \quad n = 1, 2, \dots. \tag{2.13}$$

Now, we can deduce by induction, that $|b_n| \leq C_n e^{K(n-1)}$ for $n \geq 1$, where $K : \mathbb{N} \rightarrow \mathbb{R}$ is defined in

Lemma 2.1. In fact $|b_1| = |\eta| = C_1$. For inductive proof we assume that $|b_j| \leq C_j e^{K(j-1)}$, $j \leq m$. From (2.9) and Lemma 2.1

$$\begin{aligned} |b_{n+1}| &\leq \frac{L}{|1-\alpha^n|} \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} |b_{i+1}| |b_j| |b_k| |b_{n-i-j-k}| \\ &\leq \frac{L}{|1-\alpha^n|} \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} C_{i+1} C_j C_k C_{n-i-j-k} e^{K(i)+K(j-1)+K(k-1)+K(n-i-j-k-1)}. \end{aligned}$$

Note that

$$\begin{aligned} &K(i) + K(j-1) + K(k-1) + K(n-i-j-k-1) \\ &\leq K(n-3) \leq K(m-1) \leq \log|\alpha^n - 1| + K(m). \end{aligned}$$

Then

$$|b_{n+1}| \leq e^{K(m)} \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} C_{i+1} C_j C_k C_{n-i-j-k} = C_{n+1} e^{K(n)}.$$

as desired. Note that $K(n) \leq n(B(\theta) + \gamma)$ for some universal constant $\gamma > 0$. Then

$$|b_n| \leq T^n e^{(n-1)(B(\theta)+\gamma)},$$

that is

$$\limsup_{n \rightarrow \infty} (|b_n|)^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} (T^n e^{(n-1)(B(\theta)+\gamma)})^{\frac{1}{n}} = T e^{B(\theta)+\gamma}.$$

This implies that the convergence radius of (2.1) is at least $(T e^{B(\theta)+\gamma})^{-1}$. This completes the proof.

In case (H_3) the constant α is not only on the unit circle in C , but also a root of unity. In such a case, the resonant case, both Diophantine condition and Brjuno condition are not satisfied.

Let $\{D_n\}_{n=0}^\infty$ be a sequence define by $D_0 = |\mu|$, $D_1 = |\eta|$ and

$$D_{n+1} = \Gamma L \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} D_{i+1} D_j D_k D_{n-i-j-k}, \quad n = 1, 2, \dots. \tag{2.14}$$

Where $\Gamma = \max\{1, |\alpha^i - 1|^{-1} : i = 1, 2, \dots, p-1\}$ and L is defined in Theorem 2.2.

Theorem 2.3 Suppose (H_3) holds and $a \neq 0, 1, b \neq 0$, and p is given as above. Let

$\{b_n\}_{n=0}^\infty$ be determined recursively by $b_0 = \mu$, $b_1 = \eta$ and

$$ab^2(n+1)(1-\alpha^n)b_{n+1} = \Phi(n, \alpha), \quad n = 1, 2, \dots. \tag{2.15}$$

where

$$\Phi(n, \alpha) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} (i+1)\alpha^j (\alpha^j - a)(\alpha^k - a)(\alpha^{n-i-j-k} - a) b_{i+1} b_j b_k b_{n-i-j-k}.$$

If $\Phi(vp, \alpha) = 0$ for all $v = 1, 2, \dots$, then Eq.(1.6) has an analytic solution of the form

$$g(z) = \mu + \eta z + \sum_{n=vp+1, v \in \mathbb{N}} \xi_{vp+1} z^n + \sum_{n \neq vp+1, v \in \mathbb{N}} b_n z^n, \quad N = \{1, 2, 3, \dots\}.$$

in a neighborhood of the origin, where all ξ_{vp+1} s are arbitrary constants satisfying the inequality $|\xi_{vp+1}| \leq D_{vp+1}$ and the sequence $\{D_n\}_{n=0}^\infty$ is defined in (2.14). Otherwise, if $\Phi(vp, \alpha) \neq 0$ for some $v = 1, 2, \dots$, then Eq.(1.6) has no analytic solutions in any neighborhood of the origin.

Proof We seek a power series solution of (1.6) of the form (2.1) as in the proof of Theorem 2.1, where the equality (2.4) or (2.5) is satisfied. If $\Phi(vp, \alpha) \neq 0$ for some natural number v , then the equality in (2.15) does not hold for $n = vp$ since $1 - \alpha^{vp} = 0$. In such a circumstance Eq.(1.6) has no formal solutions.

When $\Phi(vp, \alpha) = 0$ for all natural number v , for each v the corresponding b_{vp+1} in (2.15) has infinitely many choices in C , that is, the formal series solution (2.1) defines a family of solutions with infinitely many parameters. Choose $b_{vp+1} = \xi_{vp+1}$ arbitrarily such that

$$|\xi_{vp+1}| \leq D_{vp+1}, \quad v = 1, 2, \dots, \tag{2.16}$$

Where D_{vp+1} is defined by (2.14). In what follows we prove that the formal series solution (2.1) converges in a neighborhood of the origin. Observe that $|\alpha^n - 1|_{-1} \leq \Gamma$ for $n \neq vp$. It follows from (2.15) that

$$|b_{n+1}| \leq \Gamma L \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} |b_{i+1}| |b_j| |b_k| |b_{n-i-j-k}| \tag{2.17}$$

for all $n \neq vp, v = 1, 2, \dots$.

Let

$$W(z, \mu, \eta, \Gamma, L) = \sum_{n=0}^\infty D_n z^n, \quad D_0 = |\mu|, \quad D_1 = |\eta|. \tag{2.18}$$

It is easy to check that (2.18) satisfies the implicit functional equation

$$R(z, \psi, \mu, \eta, \Gamma, L) = 0, \tag{2.19}$$

where R is defined in (2.8). Moreover, similarly to the proof of Theorem 2.1, we can prove that (2.19) has a unique analytic solution $\psi(z, \mu, \eta, \Gamma, L)$ in a neighborhood of the origin such that $\psi(0, \mu, \eta, \Gamma, L) = |\mu|$ and $\psi'_z(0, \mu, \eta, \Gamma, L) = |\eta|$. Moreover, we also have $\psi(z, \mu, \eta, \Gamma, L) = W(z, \mu, \eta, \Gamma, L)$. Thus (2.18) converges in a neighborhood of the origin. Moreover, it is easy to show that, by induction,

$$|b_n| \leq D_n, \quad n = 1, 2, \dots$$

Therefore, the series (2.1) converges in a neighborhood of the origin. This completes the proof.

III. ANALYTIC SOLUTION OF (1.2)

Theorem3.1 Under one of the conditions in Theorem 2.1-2.3, Eq.(1.5) has an analytic solution of the form $y(z) = g(\alpha g^{-1}(z))$ in a neighborhood of the number μ , where $g(z)$ is an analytic solution of (1.6).

Proof By Theorem 2.1-2.3, we may find an analytic solution $g(z)$ of the auxiliary equation (1.6) in the form of (2.1) such that $g(0) = \mu$ and $g'(0) = \eta \neq 0$. Clearly the inverse $g^{-1}(z)$ exists and is analytic in a neighborhood of $g(0) = \mu$. If we define $y(z)$ by means of $g(\alpha g^{-1}(z))$, then

$$\begin{aligned} y'(z) &= \frac{\alpha g'(\alpha g^{-1}(z))}{g'(g^{-1}(z))}, \\ b^2(a - y'(z)) &= \frac{b^2}{g'(g^{-1}(z))} [a g'(g^{-1}(z)) - \alpha g'(\alpha g^{-1}(z))] \\ &= \frac{[g(\alpha^2 g^{-1}(z)) - a g(\alpha g^{-1}(z))] [g(\alpha g^{-1}(z)) - a g(g^{-1}(z))]^2 g'(g^{-1}(z))}{g'(g^{-1}(z))} \\ &= [y(y(z)) - a y(z)] [y(z) - a z]^2. \end{aligned}$$

as required. This completes the proof.

We have shown that under the conditions of Theorem 2.1, 2.2 or 2.3, Eq.(1.5) has an analytic solution $y(z) = g(\alpha g^{-1}(z))$ in a neighborhood of the number μ , where $g(z)$ is an analytic solution of (1.6). Since the function $g(z)$ in (2.1) can be determined by (2.4), it is possible to calculate, at least in theory, the explicit form of $y(z)$, an analytic solution of (1.2), in a neighborhood of the fixed point μ of $y(z)$ by means of (1.4). However, knowing that an analytic solution of (1.2) exists, we can take an alternative route as follows. Assume that $x(z)$ is of the form

$$x(z) = x(\mu) + x'(\mu)(z - \mu) + \frac{x''(\mu)}{2!}(z - \mu)^2 + \dots; \tag{3.1}$$

we need to determine the derivatives $x^{(n)}(\mu), n = 0, 1, 2, \dots$. First of all, in view of (1.2), we have

$$x(\mu) = \frac{b}{y(\mu) - a\mu} = \frac{b}{(1-a)\mu}.$$

and

$$x'(\mu) = \frac{1}{x(a\mu + \frac{b}{x(\mu)})} = \frac{1}{x(a\mu + (1-a)\mu)} = \frac{(1-a)\mu}{b}.$$

Next by calculating the derivatives of both of (1.2), we obtain

$$x''(z) = \left(\frac{1}{x(az + \frac{b}{x(z)})}\right)' = \frac{-x'(az + \frac{b}{x(z)})\left(a - \frac{bx'(z)}{(x(z))^2}\right)}{\left(x(az + \frac{b}{x(z)})\right)^2}.$$

Thus

$$x''(\mu) = \frac{-x'(\mu)\left(a - \frac{bx'(\mu)}{(x(\mu))^2}\right)}{(x(\mu))^2} = \frac{(1-a)^3 \mu^3 [(1-a)^3 \mu^3 - ab^2]}{b^5}.$$

For convenience, we write

$$f(z) = az + \frac{b}{x(z)}, \quad x^{(n)}(\mu) := \lambda_n, \quad n = 1, 2, \dots.$$

So, $f(\mu) = \mu, f'(\mu) = a + b\lambda_2, f''(\mu) = b\lambda_3, \dots, f^{(j)}(\mu) = b\lambda_{j+1}, j \geq 2$.

By induction, we can prove

$$(x(f(z)))^{(j)} = \sum_{i=1}^j p_{ij}(f'(z), f''(z), \dots, f^{(j)}(z))x^{(i)}(f(z)),$$

where p_{ij} is a uniquely defined multivariate polynomial with non-negative coefficients. From (1.2) we have

$$x'(z)x(f(z)) = 1.$$

Calculating k th derivatives of both sides of the above equation, we obtain

$$x^{(k+1)}(z)x(f(z)) + \sum_{j=1}^k C_k^j x^{(k-j+1)}(z) \sum_{i=1}^j p_{ij}(f'(z), f''(z), \dots, f^{(j)}(z))x^{(i)}(f(z)) = 0, \\ k = 1, 2, \dots.$$

Thus, λ_n satisfies the difference equation

$$\frac{b}{(1-a)\mu} \lambda_{k+1} + \sum_{j=1}^k C_k^j x^{(k-j+1)} \sum_{i=1}^j p_{ij}(a + b\lambda_2, b\lambda_3, \dots, b\lambda_{j+1}) \lambda_i = 0,$$

$$v = 1, 2, \dots$$

By means of this formula, we get the explicit form of the solution (3.1), i.e.,

$$x(z) = \frac{b}{(1-a)\mu} + \frac{(1-a)\mu}{b}(z-\mu) + \frac{(1-a)^3\mu^3[(1-a)^3\mu^3 - ab^2]}{2b^5}(z-\mu)^2 + \sum_{n=3}^{\infty} \frac{\lambda_n}{n!}(z-\mu)^n.$$

ACKNOWLEDGMENTS

This work was supported by the Research Fund of Binzhou University (No. BZXYL1203 and No. 2010Y09) and the Subject Construction Funds of Mathematics in Binzhou University.

REFERENCES

- [1] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, 1977..
- [2] R. Bellman, K.Cooke, *Differential-Difference Equations*, Academic Press, 1963.
- [3] E. Eder, The functional differential equation $x'(t) = x(x(t))$ [J]. *Differential Equations* 54 (1984): 390-400.
- [4] Z. Jackiewicz, Existence and uniqueness of solutions of neutral delay-differential equations with state dependent delays[J], *Funkcial. Ekvac.* 30 (1987) :9-17.
- [5] F. Qiu, H.Z. Liu, Analytic solutions of a first order iterative differential equation[J]. *Demonstratio Mathematica* 1 (2006): 81-90.
- [6] F. Qiu, M. Su, Analytic solutions of a first order iterative differential equation[J]. *Journal of Liaocheng University.* 18(4)(2005): 18-23.
- [7] J. Si, W. Li, S. Cheng, Analytic solutions of an iterative functional differential equation[J]. *Comput. Math. Appl.* 33(6) (1997) : 47-51
- [8] J. Si, S. Cheng, Note on an iterative functional differential equation[J]. *Demonstratio Math.* 31 (3) (1998) : 599-614.
- [9] X. Wang, J. Si, Analytic solutions of an iterative functional differential equation[J]. *Math. Anal. Appl.* 262 (2001) : 490-498.
- [10] J. Si, W. Zhang, Analytic solutions of a class of iterative functional differential equations[J]. *Comput. Appl. Math.* 162 (2004) :467-481.
- [11] J. Si, X. Wang, S. Cheng, Analytic solutions of a functional differential equation with a state derivative dependent delay[J]. *Aequationes Math.* 57 (1999): 75-86.
- [12] J. Si, S. Cheng, Smooth solutions of a nonhomogeneous iterative functional differential equation[J]. *Proc. Roy. Soc. Edinburgh Sect. A* 128 (1998) : 821-831.
- [13] J. Si, Sui-Sun Cheng, Analytic solutions of a functional differential equation with state dependent argument[J]. *Taiwanese journal of mathematics*, 1997, 1(4):471-480.
- [14] A.D. Brjuno, Analytic form of differential equations[J]. *Trans. Moscow Math. Soc.* 25 (1971) : 131-288.
- [15] T. Carletti, S. Marmi, Linearization of analytic and non-analytic germs of diffeomorphisms of $(C, 0)$ [J]. *Bull. Soc. Math. France* , 128 (2000) : 69-85.
- [16] A.M. Davie, The critical function for the semistandard map[J]. *Nonlinearity* 7 (1994) : 219-229.

Jing Liu

Jing Liu is an associate professor at the Department of Mathematics in Binzhou University, Shandong, PRC. Her research interests include analytic solution and differential equation theory.