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Relationship between cores of targets and Euclidean controllability for single-delay neutral control systems

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ABSTRACT : *This article established necessary and sufficient conditions for the compactness of cores of Euclidean targets and Euclidean controllability of single-delay linear neutral control systems.*
The proofs relied on the notion of asymptotic directions and other convex set properties as well as the independent results on cores of targets and Euclidean controllability in Ukwu [1, 2,3] respectively.

KEYWORDS: *Asymptotic, Core, Controllability, Euclidean, Targets.*

I. INTRODUCTION

The relationship between cores of targets and Euclidean controllability was introduced by Hajek [4], who examined the system $\dot{x} = Ax - p,\ p(t) \in P,\ x(\text{end}) \in T,$ where A is an $n \times n$ coefficient matrix, $P,$ the constraint set a compact convex non-void subset of the real n-dimensional Euclidean space $\mathbf{R}^n$, the target set, T, a closed convex non-void subset of $\mathbf{R}^n$, and $p: I \to P,$ admissible controls on I, a subset of $\mathbf{R}^+ = [0, \infty).$ By exploiting the analyticity and non-singularity of the fundamental matrices of the associated homogeneous system $\dot{x} = Ax,$ and using the notion of asymptotic directions and other convex set properties, he established that $\operatorname{core}(T)$ is bounded if and only if $\operatorname{rank}\left[M^T, A^T M^T, \cdots, \left(A^T\right)^{n-1} M^T \right] = n,$ for some $m \times n$ constant matrix, M. He indicated that the closedness of $\operatorname{core}(T)$ could be achieved by using a weak compactness argument.
[5]Extended Hajek's results to a delay control system, with the major contribution being the varying of the technique for the boundedness of $\operatorname{core}(T)$ due to the singularity of the solution matrices and certain other properties of such matrices.This paper extends further the relationship between cores of targets and Euclidean controllability to single-delay linear neutral systems by leaveraging on the results in [1, 2, 3].
Consider the autonomous linear differential – difference control system of neutral type:

$$\dot{x}(t) = A_{-1}\dot{x}(t-h) + A_0 x(t) + A_1 x(t-h) + B\,u(t);\ t \ge 0 \tag{1}$$

$$x(t) = \phi(t),\ t \in [-h, 0],\ h > 0 \tag{2}$$

where A_{-1}, A_0, A_1 are $n \times n$ constant matrices with real entries and B is an $n \times m$ constant matrix with the real entries. The initial function ϕ is in $C\left([-h, 0], \mathbf{R}^n\right)$ equipped with sup norm. The control u is in $L_\infty\left([0, t_1], \mathbf{R}^n\right)$. Such controls will be called admissible controls. $x(t), x(t-h) \in \mathbf{R}^n$ for $t \in [0, t_1]$. If $x \in C\left([-h, t_1], \mathbf{R}^n\right)$, then for $t \in [0, t_1]$ we define $x_t \in C\left([-h, 0], \mathbf{R}^n\right)$ by $x_t(s) = x(t+s),\ s \in [-h, 0]$

II. EXISTENCE, UNIQUENESS AND REPRESENTATION OF SOLUTIONS

If $A_{-1} \ne 0$ and ϕ is continuously differentiable on $[-h, 0]$, then there exists a unique function $x: [-h, \infty)$ which coincides with ϕ on $[-h, 0]$, is continuously differentiable and satisfies (1) except possibly at the points $jh;\ j = 0, 1, 2, \ldots$. This solution x can have no more derivatives than ϕ and continuously differentiable if and only if the relation:

$$\dot{\phi}(0) = A_{-1}\,\dot{\phi}(-h) + A_0\phi(0) + A_1\,\phi(-h) + Bu(0) \tag{3}$$

is satisfied. See Bellman and Cooke [6] and theorem 7.1 in Dauer and Gahl [7] for complete discussion on existence, uniqueness and representations of solutions of (1).

2.1 Preliminaries on the Partial Derivatives $\dfrac{\partial^k X(\tau,t)}{\partial \tau^k}, k = 0,1,\cdots$

Let $t, \tau \in [0,\ t_1]$ for fixed t, let $\tau \to X(\tau,t)$ be the unique function satisfying the matrix differential equation:

$$\frac{\partial}{\partial \tau} X(\tau,t) = \frac{\partial}{\partial \tau} X(\tau+h,t) A_{-1} - X(\tau,t) A_0 - X(\tau+h,t) A_1 \tag{4}$$

$0 < \tau < t, \tau \neq t - jh;\ j = 0,1,...,$ subject to:

$$X(\tau,t) = \begin{cases} I_n;\ \tau = t \\ 0,\ \ \tau > t \end{cases} \tag{5}$$

By Tadmore ([8], p. 80), $\tau \to X(\tau,t)$ is analytic on $(t-(j+1)h,\ t-jh),\ j = 0,1,....$ and hence $\tau \to X(\tau,t)$ is C^∞ on these intervals.

The left and right-hand limits of $\tau \to X(\tau,t)$ exist at $\tau = t - jh$, so that $X(\tau,t)$ is of bounded variation on each compact interval. Cf. Banks and Jacobs [9], Banks and Kent [10] and Hale [11]. We maintain the notation $X^{(k)}(\tau,t),\ \Delta X^{(k)}(\tau,t)$ as in [1].

III. THEOREM RELATING CORES OF TARGETS AND EUCLIDEAN CONTROLLABILITY FOR SYSTEM (1)

Consider the control system (1) with its standing hypotheses. Let the target set H be of the form $H = L + D,$ where $L = \{x \in \mathbb{R}^n : M\,x = 0\}$ for some $m \times n$ matrix $M,$ and some bounded, convex subset D of $\mathbb{R}^n$ with $0 \in D$. Assume that $0 \in U,$ and that $0 \in H$. Then core(H) is compact if and only if the system:

$$\dot{x}(t) = A_{-1}^T \dot{x}(t-h) + A_0^T x(t) + A_1^T x(t-h) + M^T u(t) \tag{6}$$

is Euclidean controllable.

<u>Proof</u>

Let a be an asymptotic direction of core(H). Let $t \to Y(t,\tau) = Y(t-\tau)$ be a solution matrix of the free part of (1) such that $Y(0) = I_n$, the identity matrix of order n. Then by lemma 4.2 of [1], we deduce that $Y(t)a$ is an asymptotic direction of H for $t \geq 0.$ By lemma 2.6 of [1],

$Y(t)a \in L,$ for all $t \geq 0.$ It follows from the definition of L in the hypotheses of theorem 4.4 that:

$$MY(t)a = 0 \tag{7}$$

for each $a \in O^+(\text{core}(H))$ and $t \geq 0,$ where $O^+(K)$ is the set of asymptotic directions of the convex set $K,$ as defined in [1]. Take the transpose of both sides of (7) to get $a^T Y^T(t) M^T = 0.$ By lemma 4and theorem 4.1 of [1], core(H) is a nonvoid convex subset of $\mathbf{R}^n$. By lemma 2.5 of [1], core(H) is bounded if and only if 0 is its only asymptotic direction.

Claim: $Y(t,\tau)$ satisfies $\dfrac{\partial}{\partial\tau} Y^T(t,\tau) = \dfrac{\partial}{\partial\tau} Y^T(t,\tau+h) A_{-1} - Y^T(t,\tau) A_0^T - Y^T(t,\tau+h) A_1^T$

Proof of the claim: Since $t \to Y(t,\tau)$ is a solution matrix of the free part of (1), it follows that

$\dfrac{\partial}{\partial t} Y(t,\tau) = A_{-1} \dfrac{\partial}{\partial t} Y(t,\tau+h) + A_0 Y(t,\tau) + A_1 Y(t,\tau+h)$, from which transposition yields

(i) $\frac{\partial}{\partial t}Y^T(t,\tau)=\frac{\partial}{\partial t}Y^T(t,\tau+h)A_{-1}^T+Y^T(t,\tau)A_0^T+Y^T(t,\tau+h)A_1^T$

Set $\hat{X}(\tau,t)=Y^T(t,\tau)$. Then $\hat{X}(\tau,\ t)=\hat{X}(\tau-t)$ and $Y^T(t,\tau)=\hat{X}(\tau-t)$. Consequently,

$$\frac{\partial}{\partial t}Y^T(t,\tau)=\frac{\partial}{\partial(\tau-t)}\hat{X}(\tau-t)\frac{\partial(\tau-t)}{\partial t}=-\frac{\partial}{\partial(\tau-t)}\hat{X}(\tau-t)$$

$$\frac{\partial}{\partial t}Y^T(t,\tau+h)=\frac{\partial}{\partial(\tau+h-t)}\hat{X}(\tau+h-t)\frac{\partial(\tau+h-t)}{\partial t}=-\frac{\partial}{\partial(\tau+h-t)}\hat{X}(\tau+h-t)$$

Clearly, $\frac{\partial}{\partial\tau}Y^T(t,\tau)=\frac{\partial}{\partial(\tau-t)}\hat{X}(\tau-t)\frac{\partial(\tau-t)}{\partial\tau}=\frac{\partial}{\partial(\tau-t)}\hat{X}(t-\tau)$

$$\Rightarrow \text{(ii)}\quad \frac{\partial\hat{X}(\tau,t)}{\partial\tau}=\frac{\partial}{\partial\tau}Y^T(t,\tau)=-\frac{\partial}{\partial t}Y^T(t,\tau).$$

$$\frac{\partial}{\partial\tau}Y^T(t,\tau+h)=\frac{\partial}{\partial(\tau+h-t)}\hat{X}(\tau+h-t)\frac{\partial(\tau+h-t)}{\partial\tau}=\frac{\partial}{\partial(\tau+h-t)}\hat{X}(\tau+h-t)$$

$$\Rightarrow \text{(iii)}\quad \frac{\partial\hat{X}(\tau+h,t)}{\partial\tau}=\frac{\partial}{\partial\tau}Y^T(t,\tau+h)=-\frac{\partial}{\partial t}Y^T(t,\tau+h).$$

(i), (ii), and (iii) yield

$$\frac{\partial}{\partial\tau}\hat{X}(\tau,t)=\frac{\partial}{\partial\tau}\hat{X}(\tau+h,\ t)A_{-1}^T-\hat{X}(\tau,t)A_0^T-\hat{X}(\tau+h,t)A_1^T$$

$\frac{\partial}{\partial\tau}Y^T(t,\tau)=\frac{\partial}{\partial\tau}Y^T(t,\tau+h)A_{-1}^T-Y^T(t,\tau)A_0^T-Y^T(t,\tau+h)A_1^T$, proving the claim.

Let $\dot{x}(t)=A_{-1}^T\dot{x}(t-h)+A_0^Tx(t)+A_1^Tx(t-h)+M^Tu(t)$ be Euclidean controllable on $[0,\ t_1]$, for some $t_1>0.$ By the above claim and the proof of theorem 2 of [2], this is equivalent to requiring that the relation $a^TY^T(t,\tau)M^T\ =\ 0,$ implies $a=0,$ for $t\in[\max\{0,\ \tau\},\ t_1].$ Therefore the relation $a^TY^T(t)M^T\ =\ 0$ implies $a=0,$ for $t\in[0,\ t_1].$ But a is an arbitrary asymptotic direction of the nonvoid convex set, core(H). Hence $O^+(\text{core}(H))=\{0\}$. We invoke lemma 2.5 to conclude that core(H) is bounded, since O is its only asymptotic direction.

It is already established in theorem 4.1 of [1], that core(H) is closed. Therefore core(H) is compact.

Conversely let core(H) be compact. Then $O^+(\text{core}(H))=\{0\}$, by lemma 2.5. Suppose (6) is not Euclidean controllable. Then by theorem 2 of [2], there exists $c\in\mathbf{R}^n, c\neq 0,$ such that $c^TY^T(t)M^T=0,\quad\forall t\in[0,\ t_1].$ Hence by transposition, $MY(t)c=0$. Consequently, $Y(t)c\in L=O^+(H)$. We appeal to lemma 4.2 of [1] to deduce that $c\in O^+(\text{core}(H)), c\neq 0.$. This contradicts the compactness of core , by lemma 2.5 of [1]. Hence the supposition is false. Therefore (6) is Euclidean controllable given that core(H) is compact. This concludes the proof of the theorem.

Remarks:

If $A_1=A_{-1}=0,$ then the system (6) reduces to $\dot{x}(t)=A_0^Tx(t)+M^Tu(t)$, in which case the fundamental matrix $Y^T(t)$ of $\dot{x}(t)=A_0^Tx(t)$ is analytic, nonsingular and equals $e^{A_0^Tt}$. This particular case $(A_1=A_{-1}=0)$ forces $j=0$ in the formula for $Q_k(jh)$ and $Q_k(0),$ for $s\neq 0.$ But $k\geq 0\Rightarrow Q_k(0)=A_0^k$, by lemma 2.4 of [3]. We then conclude from theorem 4.4of [1] that core(H) is compact if and if only rank$\left[M^T,A_0^TM^T,\ldots(A_0^T)^{n-1}M^T\right]=n$. This coincides with Hajek's result.

IV. ANALYSIS OF SYSTEM (1) AND AN ECONOMIC INTERPRETATION OF THEOREM IN 3.

The target H could represent the prescribed and desired level of stock of capital assets. The lag h represents the lead times or periods of gestation for investment in capital stock and $P = \{Bu: u \in \Omega\}$ denotes the investment capacity of the given firm. Given $u \in \Omega$, $u = (u_1, u_2, ..., u_m)^T$, it is clear that $u_1, u_2, \cdots, u_m$ are investment strategies. Strategies could be at positive, negative or zero levels. In other words strategies are unsigned or unrestricted in sign. Write $B = (b_1, b_2, \cdots, b_m)$, where b_i is a column n-vector and $i \in \{1, 2, \cdots, m\}$. Then B is a matrix of fixed stock and $Bu = \sum_{i=1}^{m} b_i u_i$, , where $b_i u_i$ denotes the availability of stock b_i to be invested at a level u_i; in other words, u_i units of stock b_i are available for investment. Thus, for a given strategy $u = (u_1, u_2, \cdots, u_m)^T$ the elements $u_1, u_2, \cdots, u_m$ represent investment coefficients of the stock $b_1, b_2, \cdots, b_m$ respectively. Hence the definition of P as the power of investment available to the firm is justified.

The values $t, x(t)$ denote respectively the current time and the current stock of capital assets, given an initial endowment ϕ. The values $x(t-h)$ denotes respectively available stock of capital assets h time units time units before, while $\dot{x}(t)$ and $\dot{x}(t-h)$ denote respectively the current rate of change of capital stock and the rate of change of capital stock h time- period before. A_0 is the matrix of coefficients of stock currently available, A_1 is the matrix of coefficients of stock available $h.$ time units; A_{-1} is the matrix of coefficients of investments an h time- period before.

Therefore system (1) could be a dynamic model of the fluctuations of capital stock, where the current rate of change of stock depends partly on the following:

[1] Currently available capital stock,
[2] Capital stock available h time units before
[3] Rate of change of capital stock h time units before and
[4] The investment capacity of the firm.

Theorem 3 says that the prescribed target, H is achievable if and only if

$$\operatorname{rank} \hat{Q}_n(t_1) = n,$$

for some $t_1 > 0$, where

$$Q_k(s) = A_{-1}^T Q_k(s-h) + A_0^T Q_{k-1}(s) + A_1^T Q_{k-1}(s-h); k \in \{0, 1, \cdots\}, s \in \{0, h, \cdots\},$$

with initial conditions.

$$Q_0(0) = I_n, Q_k(s) = 0, \text{for } \min\{k, s\} < 0, \text{ and}$$

$$\hat{Q}_n(t_1) = \left[Q_0(s)M^T, Q_1(s)M^T, \cdots, Q_{n-1}(s)M^T, s \in [0, t_1) \cap \{0, h, \cdots\}\right].$$

In other words, it is possible to control any initial endowment ϕ to a prescribed level of growth of capital stock, subject to its constraints by a careful implementation of investment strategies. However, the achieved growth level can only be sustained for small (compact) levels of initial values; otherwise there is no guarantee that the growth will not be reversed. Compactness of core (H) means that the initial capital assets need not be too big.

V. CONCLUSION

This paper relied on the results in [1], [2] and [3] to establish necessary and sufficient conditions for the Euclidean controllability of system (1) in terms of the compactness of cores of targets of a corresponding transposed system. The result exploited the specified structure of the target set amenable to the application of the notion of asymptotic directions and other convex set properties, as well as the special relationship between the solution matrices of the uncontrolled part of system (1) and the associated control index matrices defined in system (3). In the sequel the paper provided an analysis of system (1) and an economic interpretation of theorem

3.1. The ideas exposed in this paper can be tapped to extend the result and economic interpretations to associated perturbed and more general systems.

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