International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 2Issue 4 ‖ April. 2014 ‖ PP-33-41

Estimation of Parameters of Bivariate Bilognormal Distribution

[1,] Dr. Parag B Shah, [2,] Dr. C. D. Bhavsar
[1,] Dept. of Statistics, H.L.College of Commerce, Ahmedabad-380009, GUJARAT, INDIA
[2,] Dept. of Statistics, Gujarat University, Ahmedabad-380009, GUJARAT, INDIA

ABSTRACT: *In this paper we define bivariate bilognormal distribution with seven parameters ($\mu_1,\mu_2,\sigma_{11},\sigma_{12},\sigma_{21},\sigma_{22},\rho$). Bivariate bilognormal distribution with five parameters (μ_1, μ_2, σ_{11}, σ_{21}, ρ) is deduced from seven parameters distribution when the deviations of two component normal distributions of two variables x and y are proportional to each other. We try to derive some elementary properties of bivariate bilognormal distribution and estimate the parameters using method of moments and method of maximum likelihood.*

KEYWORDS: *Bivariate bilognormal distribution, Method of moments, Method of maximum likelihood, Concomitant observations.*

I. INTRODUCTION:

Various researchers such as Galton (1879) and Wal Chuan-yi (1966), have shown the suitability of the lognormal distribution as a limiting distribution of order Statistics under certain conditions. Nabar and Desmukh (2002) proposed Bilognormal distribution to model income distribution and life time distribution. We define Bivariate Bilognormal distribution with seven parameters $\mu_1,\mu_2,\sigma_{11},\sigma_{12},\sigma_{21},\sigma_{22}$ and ρ as follows :

$$f(x,y)\begin{cases} Ko(xy)^{-1}\exp\left[\dfrac{-1}{2(1-\rho^2)}\left\{\left(\dfrac{\log x-\mu_1}{\sigma_{11}}\right)^2-2\rho\left(\dfrac{\log x-\mu_1}{\sigma_{11}}\right)^2\left(\dfrac{\log y-\mu_2}{\sigma_{21}}\right)+\left(\dfrac{\log y-\mu_2}{\sigma_{21}}\right)^2\right\}\right] \\ \qquad\qquad \log x\le\mu_1,\log y\le\mu_2 \\ Ko(xy)^{-1}\exp\left[\dfrac{-1}{2(1-\rho^2)}\left\{\left(\dfrac{\log x-\mu_1}{\sigma_{11}}\right)^2-2\rho\left(\dfrac{\log x-\mu_1}{\sigma_{11}}\right)^2\left(\dfrac{\log y-\mu_2}{\sigma_{22}}\right)+\left(\dfrac{\log y-\mu_2}{\sigma_{22}}\right)^2\right\}\right] \\ \qquad\qquad \log x\le\mu_1,\log y>\mu_2 \\ Ko(xy)^{-1}\exp\left[\dfrac{-1}{2(1-\rho^2)}\left\{\left(\dfrac{\log x-\mu_1}{\sigma_{12}}\right)^2-2\rho\left(\dfrac{\log x-\mu_1}{\sigma_{12}}\right)^2\left(\dfrac{\log y-\mu_2}{\sigma_{21}}\right)+\left(\dfrac{\log y-\mu_2}{\sigma_{21}}\right)^2\right\}\right] \\ \qquad\qquad \log x>\mu_1,\log y\le\mu_2 \\ Ko(xy)^{-1}\exp\left[\dfrac{-1}{2(1-\rho^2)}\left\{\left(\dfrac{\log x-\mu_1}{\sigma_{12}}\right)^2-2\rho\left(\dfrac{\log x-\mu_1}{\sigma_{12}}\right)^2\left(\dfrac{\log y-\mu_2}{\sigma_{22}}\right)+\left(\dfrac{\log y-\mu_2}{\sigma_{22}}\right)^2\right\}\right] \\ \qquad\qquad \log x>\mu_1,\log y>\mu_2 \end{cases}$$

$$\text{where } Ko=\frac{2}{\pi}\left[(\sigma_{11}+\sigma_{12})(\sigma_{21}+\sigma_{22})\sqrt{1-\rho^2}\right]^{-1}$$

…….(1.1)

Now, (1.1) can be rewritten as

$$f(z_{11}, z_{21}, z_{12}, z_{22}) = \begin{cases} f(z_{11}, z_{21}) & , (z_{11}, z_{21}) \in A_1 \} \\ f(z_{11}, z_{22}) & , (z_{11}, z_{22}) \in A_2 \} \\ f(z_{12}, z_{21}) & , (z_{12}, z_{21}) \in A_3 \} \\ f(z_{12}, z_{22}) & , (z_{12}, z_{22}) \in A_4 \} \end{cases}$$

where $A_1 = \left\{ (z_{11}, z_{21}) : z_{11} \le 0,\ z_{21} \le 0 \right\}$, $A_2 = \left\{ (z_{11}, z_{21}) : z_{11} \le 0,\ z_{22} > 0 \right\}$

$A_3 = \left\{ (z_{11}, z_{21}) : z_{12} > 0,\ z_{21} \le 0 \right\}$, $A_4 = \left\{ (z_{12}, z_{22}) : z_{12} > 0,\ z_{22} > 0 \right\}$

and

$$z_{11} = \frac{\log x - \mu_1}{\sigma_{11}},\ z_{21} = \frac{\log y - \mu_2}{\sigma_{21}},\ z_{12} = \frac{\log x - \mu_1}{\sigma_{12}} \text{ and } z_{22} = \frac{\log y - \mu_2}{\sigma_{22}}$$

$$f_1(z_{11}, z_{21}) = \frac{K_0}{\sigma_{11}\sigma_{21}} \exp\left[-\frac{1}{2}\left(1-\rho^2\right)^{-1} \left\{ z_{11}^2 - 2\rho z_{11} z_{21} + z_{21}^2 \right\} \right]$$

$$f_1(z_{11}, z_{22}) = \frac{K_0}{\sigma_{11}\sigma_{22}} \exp\left[-\frac{1}{2}\left(1-\rho^2\right)^{-1} \left\{ z_{11}^2 - 2\rho z_{11} z_{22} + z_{22}^2 \right\} \right]$$

$$f_1(z_{12}, z_{21}) = \frac{K_0}{\sigma_{12}\sigma_{21}} \exp\left[-\frac{1}{2}\left(1-\rho^2\right)^{-1} \left\{ z_{12}^2 - 2\rho z_{12} z_{21} + z_{21}^2 \right\} \right]$$

$$f_1(z_{12}, z_{22}) = \frac{K_0}{\sigma_{12}\sigma_{22}} \exp\left[-\frac{1}{2}\left(1-\rho^2\right)^{-1} \left\{ z_{12}^2 - 2\rho z_{12} z_{22} + z_{22}^2 \right\} \right]$$

$$k_0 = \frac{2}{\pi}\left[(\sigma_{11} + \sigma_{11})(\sigma_{21} + \sigma_{22})\sqrt{1-\rho^2} \right]^{-1}$$

......(1.2)

Bivariate Bilognormal distribution with five parameters $\mu_1, \mu_2, \sigma_{11}, \sigma_{21}$ and ρ when $\sigma_{12} = k_1\sigma_{12}$ and $\sigma_{22} = k_2\sigma_{21}$ can be deduced from (1.2) as follows :

$$f(z_{11}, z_{21}, z_{12}, z_{22}) = \begin{cases} f(z_{11}, z_{21}) & , (z_{11}, z_{21}) \in A_1 \} \\ f(z_{11}, z_{22}) & , (z_{11}, z_{22}) \in A_2 \} \\ f(z_{12}, z_{21}) & , (z_{12}, z_{21}) \in A_3 \} \\ f(z_{12}, z_{22}) & , (z_{12}, z_{22}) \in A_4 \} \end{cases}$$

where

$$f_1(z_{11}, z_{21}) = \frac{K_0}{\sigma_{11}\sigma_{21}} \exp\left[-\frac{1}{2}\left(1-\rho^2\right)^{-1} \left\{ z_{11}^2 - 2\rho z_{11} z_{21} + z_{21}^2 \right\} \right]$$

$$f_2(z_{11}, z_{21}) = \frac{K_0}{\sigma_{11} k_2 \sigma_{21}} \exp\left[-\frac{1}{2}\left(1-\rho^2\right)^{-1} \left\{ z_{11}^2 - 2\rho z_{11} z_{22} + \left(\frac{z_{21}}{k_2}\right)^2 \right\} \right]$$

$$f_3(z_{12}, z_{21}) = \frac{K_0}{k_1 \sigma_{11} \sigma_{21}} \exp\left[-\frac{1}{2}\left(1-\rho^2\right)^{-1} \left\{ \left(\frac{z_{11}}{k_1}\right)^2 - 2\rho \frac{z_{11}}{k_1} z_{21} + z_{21}^2 \right\} \right]$$

$$f_4(z_{12}, z_{21}) = \frac{K_0}{k_1 \sigma_{11} k_2 \sigma_{21}} \exp\left[-\frac{1}{2}\left(1-\rho^2\right)^{-1} \left\{ \left(\frac{z_{11}}{k_1}\right)^2 - 2\rho \frac{z_{11}}{k_1} . \frac{z_{21}}{k_2} + \left(\frac{z_{21}}{k_2}\right)^2 \right\} \right]$$

$$k_0 = \frac{2}{\pi}\left[\sigma_{11}\sigma_{21}(1+k_1)(1+k_2)\sqrt{1-\rho^2}\right]^{-1}$$

$$z_{12} = \frac{z_{11}}{k_1},\ z_{22} = \frac{z_{21}}{k_2}$$

.......(1.3)

In section – 2, we derive elementary properties of Bivariate Bilognormal distribution.
In section 3 & 4, we estimate the parameters by the method of moments and method of M.L.E. respectively.

II. ELEMENTARY PROPERTIES:-

The cumulative distribution function of the Bivariate Bilognormal distribution
BVBILN (μ_1, μ_2, σ_{11}, σ_{12}, σ_{21}, σ_{22}, ρ) is given by

$$F(z_{11}, z_{21}, z_{12}, z_{22}) = \begin{cases} F_1(z_{11}, z_{21}) \ , & (z_{11}, z_{21}) \in A_1 \\ F_2(z_{11}, z_{22}) \ , & (z_{11}, z_{22}) \in A_2 \\ F_3(z_{12}, z_{21}) \ , & (z_{12}, z_{21}) \in A_3 \\ F_4(z_{12}, z_{22}) \ , & (z_{12}, z_{22}) \in A_4 \end{cases}$$

where

$$F_1(z_{11}, z_{21}) = 4C\sigma_{11}\sigma_{21}\ \Phi\left(\frac{z_{21}-\rho z_{11}}{\sqrt{1-\rho^2}}\right)\Phi(z_{11})$$

$$F_2(z_{11}, z_{22}) = 2C\sigma_{11}(\sigma_{21}-\sigma_{22})+4C\sigma_{11}\sigma_{22}\ \Phi(z_{11})\Phi\left(\frac{z_{22}-\rho z_{11}}{\sqrt{1-\rho^2}}\right)$$

$$F_3(z_{12}, z_{21}) = 2C\sigma_{11}\sigma_{21}\Phi\left(\frac{z_{21}-\rho z_{11}}{\sqrt{1-\rho^2}}\right)+2C\sigma_{12}\sigma_{21}(2\Phi(z_{12})-1)\Phi\left(\frac{z_{21}-\rho z_{12}}{\sqrt{1-\rho^2}}\right)$$

$$F_4(z_{12}, z_{22}) = 2C\sigma_{11}\left(\sigma_{21}-\sigma_{22}\right)\Phi(z_{11}) + 2C\sigma_{11}\sigma_{21}\Phi\left(\frac{z_{11}-\rho z_{21}}{\sqrt{1-\rho^2}}\right) - 2C\sigma_{12}\sigma_{21}\Phi\left(\frac{z_{12}-\rho z_{21}}{\sqrt{1-\rho^2}}\right)$$

$$+4C\sigma_{12}\sigma_{21}\ \Phi(z_{12})\Phi\left(\frac{z_{21}-\rho z_{12}}{\sqrt{1-\rho^2}}\right)+4C\sigma_{11}\sigma_{22}\ \Phi(z_{11})\Phi\left(\frac{z_{22}-\rho z_{11}}{\sqrt{1-\rho^2}}\right)$$

...... (2.1)

$C^{-1} = (\sigma_{11} + \sigma_{12})(\sigma_{21} + \sigma_{22})$ and Φ **(.)** denotes the cumulative distribution function of the standard normal distribution. The cumulative distribution function of BVBILN (μ_1, μ_2, σ_{11}, σ_{21}, ρ) can be deduced from (2.1) as follows :

$$F(z_{11}, z_{21}) = \begin{cases} F_1(z_{11}, z_{21}) \ , & (z_{11}, z_{21}) \in A_1 \\ F_2(z_{11}, z_{21}) \ , & (z_{11}, z_{21}) \in A_2 \\ F_3(z_{12}, z_{21}) \ , & (z_{12}, z_{21}) \in A_3 \\ F_4(z_{12}, z_{21}) \ , & (z_{12}, z_{21}) \in A_4 \end{cases}$$

.... (2.2)

where

$$F_1(z_{11}, z_{21}) = 4C_*\Phi(z_{11})\Phi\left(\frac{z_{21} - \rho z_{11}}{\sqrt{1-\rho^2}}\right)$$

$$F_2(z_{11}, z_{21}) = 2C_*(1-k_2) + 4C_*k_2\Phi\left(z_{11}\right)\Phi\left(\frac{\frac{z_{21}}{k_2} - \rho z_{11}}{\sqrt{1-\rho^2}}\right)$$

$$F_3(z_{12}, z_{21}) = 2C_*(1-k_1)\Phi\left(\frac{z_{21} - \rho z_{11}}{\sqrt{1-\rho^2}}\right) + 4C_*k_1k_2\Phi\left(\frac{z_{11}}{k_1}\right)\Phi\left(\frac{z_{21} - \rho\frac{z_{11}}{k_1}}{\sqrt{1-\rho^2}}\right)$$

$$F_4(z_{12}, z_{21}) = 2C_*(1-k_2)\Phi(z_{11})\Phi\left(\frac{z_{11} - \rho z_{21}}{\sqrt{1-\rho^2}}\right) - 2C_*k_1\Phi\left(\frac{\frac{z_{11}}{k_1} - \rho z_{21}}{\sqrt{1-\rho^2}}\right)$$

$$+ 4C_*k_1\Phi\left(\frac{z_{11}}{k_1}\right)\Phi\left(\frac{z_{21} - \rho\frac{z_{11}}{k_1}}{\sqrt{1-\rho^2}}\right) + 4C_*k_2\Phi\left(z_{11}\right)\Phi\left(\frac{\frac{z_{21}}{k_2} - \rho z_{11}}{\sqrt{1-\rho^2}}\right)$$

where $C_*^{-1} = \left(1+k_1\right)\left(1+k_2\right)\sigma_{11}\sigma_{21}$ and $\Phi(.)$ denotes the cumulative distribution function of the standard normal distribution.
Note that it is easy to simulate observations with density (1.1).

If we define z_1 and z_2 as standard normal variables & define

$$(z_1, z_2) = \begin{cases} \left(e^{\mu_1-\sigma_{11}|z_1|},\ e^{\mu_2-\sigma_{21}|z_2|}\right) & \text{with probability } \theta_1 \\ \left(e^{\mu_2-\sigma_{11}|z_1|},\ e^{\mu_2+\sigma_{22}|z_2|}\right) & \text{with probability } \theta_2 \\ \left(e^{\mu_1+\sigma_{12}|z_1|},\ e^{\mu_2-\sigma_{21}|z_2|}\right) & \text{with probability } \theta_3 \\ \left(e^{\mu_1+\sigma_{12}|z_1|},\ e^{\mu_2+\sigma_{22}|z_2|}\right) & \text{with probability } \theta_4 \end{cases}$$

…...(2.3)

where

$$\theta_1 = \text{Prob}\left(\log x \le \mu_1, \log y \le \mu_2\right) = \frac{\sigma_{11}\sigma_{21}}{\left(\sigma_{11}+\sigma_{12}\right)\left(\sigma_{21}+\sigma_{22}\right)}$$

$$\theta_2 = \text{Prob}\left(\log x \le \mu_1, \log y > \mu_2\right) = \frac{\sigma_{11}\sigma_{22}}{\left(\sigma_{11}+\sigma_{12}\right)\left(\sigma_{21}+\sigma_{22}\right)}$$

$$\theta_3 = \text{Prob}\left(\log x > \mu_1, \log y \le \mu_2\right) = \frac{\sigma_{12}\sigma_{21}}{\left(\sigma_{11}+\sigma_{12}\right)\left(\sigma_{21}+\sigma_{22}\right)}$$

$$\theta_4 = \text{Prob}\left(\log x > \mu_1, \log y > \mu_2\right) = \frac{\sigma_{12}\sigma_{22}}{\left(\sigma_{11}+\sigma_{12}\right)\left(\sigma_{21}+\sigma_{22}\right)}$$

.... (2.4)

then (x, y) has the BVBILN $(\mu_1, \mu_2, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \rho)$ distribution. Note that if we define (z_1, z_2) as

$$(z_1, z_2) = \begin{cases} \left(e^{\mu_1-\sigma_{11}|z_1|}, \; e^{\mu_2-\sigma_{21}|z_2|}\right) & \text{with probability } \theta_1 \\ \left(e^{\mu_1-\sigma_{11}|z_1|}, \; e^{\mu_2+k_2\sigma_{21}|z_2|}\right) & \text{with probability } \theta_2 \\ \left(e^{\mu_1+k_1\sigma_{11}|z_1|}, \; e^{\mu_2-\sigma_{21}|z_2|}\right) & \text{with probability } \theta_3 \\ \left(e^{\mu_1+k_1\sigma_{11}|z_1|}, \; e^{\mu_2+k_2\sigma_{21}|z_2|}\right) & \text{with probability } \theta_4 \end{cases}$$

where $\theta_1^{-1} = \left(1+k_1\right)\left(1+k_2\right)$, $\theta_2^{-1} = \left(1+k_1\right)\left(1+k_2\right)/k_2$,

$\theta_3^{-1} = \left(1+k_1\right)\left(1+k_2\right)/k_1$ and $\theta_4^{-1} = \left(1+k_1\right)\left(1+k_2\right)/k_1k_2$

.... (2.5)

then (x, y) has the BVBILN $(\mu_1, \mu_2, \sigma_{11}, \sigma_{21}, \rho)$ distribution.

The t^{th} row moments of BVBILN $(\mu_1, \mu_2, \sigma_{11}, \sigma_{21}, \rho)$ distribution about zero are given by

$$\mu'_{r0} = E\left(x^r\right) = \frac{2e^{r\mu_1}}{(1+k_1)}\left\{e^{\frac{1}{2}r^2\sigma_{11}^2}\Phi\left(-\sigma_{11}r\right) + k_1 e^{\frac{1}{2}r^2\sigma_{11}^2}\left(1-\Phi\left(-\sigma_{11}k_1r\right)\right)\right\}$$

........(2.6)

and

$$\mu'_{0s} = E\left(y^t\right) = \frac{2e^{s\mu_2}}{\left(1+k_2\right)}\left\{e^{\frac{1}{2}s^2\sigma_{21}^2}\Phi\left(-\sigma_{21}s\right) + k_2 e^{\frac{1}{2}s^2k_2^2\sigma_{21}^2}\left(1-\Phi\left(-\sigma_{21}k_2s\right)\right)\right\}$$

....(2.7)

The moments of $(\log x, \log y)$ about (μ_1, μ_2) are given by

....(2.8)

$$\mu^*_{0s} = E(\log y - \mu_2)^s = \frac{2^{\left(\frac{s}{2}\right)}}{\sqrt{\pi}} \frac{\sigma_{21}^{\ s}\ \Gamma\left(\frac{r+1}{2}\right)}{(1+k_2)} \left\{(-1)^s + k_2^{\ s+1}\right\}$$

....(2.9)

Putting r = 1, 2, 3,…. in **(2.8)** and **(2.9)** we get

$$\mu^*_{10} = \mathrm{E}(\log x - \mu_1) = \sqrt{\frac{2}{\pi}}\, \sigma_{11}(k_1 - 1),$$

$$\mu^*_{20} = \mathrm{E}(\log x - \mu_1)^2 = \sigma^2_{11}\left(k_1^2 - k_1 + 1\right)$$

$$\mu^*_{30} = \mathrm{E}(\log x - \mu_1)^3 = 2\sqrt{\frac{2}{\pi}}\sigma^3_{11}(k_1 - 1)\left(k_1^2 + 1\right)$$

$$\mu^*_{01} = \mathrm{E}(\log y - \mu_2) = \sqrt{\frac{2}{\pi}}\sigma_{21}(k_2 - 1)$$

$$\mu^*_{02} = \mathrm{E}(\log x - \mu_2)^2 = \sigma^2_{21}\left(k_2^2 - k_2 + 1\right)$$

$$\mu^*_{03} = \mathrm{E}(\log y - \mu_2)^3 = 2\sqrt{\frac{2}{\pi}}\sigma^3_{21}(k_2 - 1)\left(k_2^2 + 1\right)$$

The central moments of log x and log y of order two and three are given by

$$\mu_{20} = \mu^*_{20} - \left(\mu^*_{10}\right)^2$$

$$= \sigma^2_{11}\left((k_1 - 1)^2 \left(1 - \frac{2}{\pi}\right) + k_1\right)$$

$$\mu_{30} = \mu^*_{30} - 3\mu^*_{20}\mu^*_{10} + 2\mu^{*3}_{10}$$

$$= \sqrt{\frac{2}{\pi}}\, \sigma^3_{11}(k_1 - 1)\left[\left((k_1 - 1)^2 \frac{4}{\pi}(k_1 - 1) + k_1\right)\right]$$

and

$$\mu_{02} = \mu^*_{02} - \left(\mu^*_{01}\right)^2$$

$$= \sigma^2_{21}\left[(k_2 - 1)^2 \left(1 - \frac{2}{\pi}\right) + k_2\right]$$

$$\mu_{03} = \mu^*_{03} - 3\mu^*_{02}\mu^*_{01} + 2\mu^{*3}_{01}$$

$$= \sqrt{\frac{2}{\pi}}\, \sigma^3_{21}(k_2 - 1)\left[(k_2 - 1)^2 \frac{4}{\pi}(k_2 - 1) + k_2\right]$$

III. ESTIMATION BY THE METHOD OF MOMENTS:-

It is difficult to obtain the estimators of the parameters μ_1, μ_2, σ_{11}, σ_{21}, and ρ using the row moments of (x, y) given in (2.6) and (2.7). If σ_{11}, σ_{21}, and

$$\widehat{\mu}_1 = \log\left\{\frac{(1+k_1)\, m'_{10}}{2\left\{e^{\frac{1}{2}\sigma_{11}^2}\Phi(-\sigma_{11}) + k_1 e^{\frac{1}{2}k_1^2\sigma_{11}^2}\left(1-\Phi(-k_1\sigma_{11})\right)\right\}}\right\}$$

........(3.1)

and

$$\widehat{\mu}_2 = \log\left\{\frac{(1+k_2)\, m'_{01}}{2\left\{e^{\frac{1}{2}\sigma_{21}^2}\Phi(-\sigma_{21}) + k_2 e^{\frac{1}{2}k_2^2\sigma_{21}^2}\left(1-\Phi(-k_2\sigma_{21})\right)\right\}}\right\}$$

..........(3.2)

where $m'_{10} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $m'_{01} = \frac{1}{n}\sum_{i=1}^{n} y_i$

However these estimators are found to be less efficient then the estimators based on the moments of (Log x, Log y), namely.

$\tilde{\mu}_1 = m_{10} - \sqrt{\frac{2}{\pi}}\sigma_{11}(k_1 - 1)$ and $\tilde{\mu}_2 = m_{01} - \sqrt{\frac{2}{\pi}}\sigma_{21}(k_2 - 1)$

..........(3.3)

where

$$m_{10} = \frac{1}{n}\sum_{i=1}^{n}\log x_i \quad \text{and} \quad m_{01} = \frac{1}{n}\sum_{i=1}^{n}\log y_i$$

It can be observed that $(\widehat{\mu}_1, \widehat{\mu}_2)$ is not satisfactory when $\sigma_{12} >> \sigma_{11}, \sigma_{22} >> \sigma_{21}$. Hence to estimate the parameters $\mu_1, \mu_2, \sigma_{11}, \sigma_{21}$, and ρ the following method is adopted. Denote the sample mean of (log x_i, log y_i) (i = 1,2,3,4........n) by (m_{10}, m_{01}) and central sample moments of order two and three by (m_{20}, m_{02}) and (m_{30}, m_{03}) respectively. Then the moment estimates satisfy the equations:-

$$m_{10} = \mu_1 + \sqrt{\frac{2}{\pi}}\sigma_{11}(k_1 - 1)$$

$$m_{10} = \mu_2 + \sqrt{\frac{2}{\pi}}\sigma_{21}(k_2 - 1)$$

$$m_{20} = \sigma_{11}^2\left(1 - \frac{2}{\pi}\right)(k_1 - 1)^2 + k_1\sigma_{11}^2$$

$$m_{02} = \sigma_{21}^2\left(1 - \frac{2}{\pi}\right)(k_2 - 1)^2 + k_2\sigma_{21}^2$$

$$m_{30} = \sqrt{\frac{2}{\pi}}\sigma_{11}^3 (k_1 - 1)\left((k_1 - 1)^2\left(\frac{4}{\pi} - 1\right) + k_1\right)$$

$$m_{03} = \sqrt{\frac{2}{\pi}}\sigma_{21}^3 (k_2 - 1)\left((k_2 - 1)^2\left(\frac{4}{\pi} - 1\right) + k_2\right)$$

The above equations are solved by the procedure given by John(1982) subject to the conditions.

$$\left| m_{30} m_{20}^{-3/2} \right| = < \left(\frac{\pi}{2} - 1 \right)^{\frac{3}{2}} - \left(\frac{\pi}{2} - 1 \right)^{\frac{1}{2}} \quad \& \quad \left| m_{03} m_{02}^{-3/2} \right| = < \left(\frac{\pi}{2} - 1 \right)^{\frac{3}{2}} - \left(\frac{\pi}{2} - 1 \right)^{\frac{1}{2}}$$

The moment estimators are given by

$$\tilde{\mu}_1 = m_{10} - B_1 \sqrt{\frac{\pi}{2}} \quad , B_1 = \tilde{\sigma}_{11} \left(k_1 - 1 \right)$$

$$\tilde{\mu}_2 = m_{01} - B_2 \sqrt{\frac{\pi}{2}} \quad , B_2 = \tilde{\sigma}_{21} \left(k_2 - 1 \right)$$

$$\tilde{\sigma}_{11} = \frac{B_1 + \sqrt{B_1^2 + 4C_1}}{2} \quad , \tilde{\sigma}_{21} = \frac{B_2 + \sqrt{B_2^2 + 4C_2}}{2}$$

and

$$\tilde{\sigma}_{12}^2 = \frac{\sqrt{B_1^2 + 4C_1} - B_1}{2} \quad , \tilde{\sigma}_{22}^2 = \frac{\sqrt{B_2^2 + 4C_2} - B_2}{2}$$

$$\tilde{\rho} = B_1 B_2 + \sqrt{B_1^2 + 4C_1} \sqrt{B_2^2 + 4C_2}$$

$$C_1 = k_1 \tilde{\sigma}_{11}^2 \quad , C_2 = k_2 \tilde{\sigma}_{11}^2$$

IV. ESTIMATION BY THE METHOD OF M.L.E:-

Let $(x_{[1:n]}, y_{[1:n]}), (x_{[1:n]}, y_{[2:n]}),\ldots,(x_{[n:n]}, y_{[n:n]})$ be a random sample of concomitant ordered paired observations of size n from BVBILN (μ_1, μ_2, σ_{11}, σ_{21}, ρ). The likelihood function is given by

Log L = Const. - n(logσ_{11} + logσ_{21} + 1/2log(1 - ρ^2))

$$-\frac{1}{2(1-\rho^2)} \left\{ \sum_1 \left(z_{11}^2 - 2\rho z_{11} z_{21} + z_{21}^2 \right) + \sum_2 \left(z_{11}^2 - 2\rho z_{11} z_{21} + z_{21}^2 / k_2^2 \right) + \sum_3 \left(\frac{z_{11}^2}{k_1^2} - \frac{2\rho z_{11}}{} \left(\frac{z_{21}}{k_1} \right) + z_{21}^2 \right) + \sum_4 \left(\frac{z_{21}^2}{k_1^2} \frac{-2\rho z_{11} z_{21}}{k_1 k_2} + \frac{z_{21}^2}{k_2^2} \right) \right.$$

……..(4.1)

$$\text{where Const.} = n \left\{ \log \left(\frac{2}{\pi} \right) - \log \left(1 + k_1 \right) - \log \left(1 + k_2 \right) - \left\{ \sum_{i=1}^{4} \left(\sum \log x_i + \sum \log y_i \right) \right\} \right\},$$

$Z_{11} = \frac{(\log x - \mu_1)}{\sigma_{11}}$ and $Z_{21} = \frac{(\log x - \mu_2)}{\sigma_{21}}$ Note that $\sum_I, \sum_2, \sum_3, \sum_4$ denotes the summation over all observations such that the conditions (i) log $x \le \mu_1$, log $y \le \mu_2$ (ii) log $x \le \mu_1$, log $y > \mu_2$, (iii) log $x > \mu_1$, log $y \le \mu_2$ and (iv) log $x > \mu_1$, log $y > \mu_2$ are satisfied by the n concomitant ordered paired observations respectively. The likelihood equations are given by

$$\frac{\partial \log L}{\partial \mu_1} = 0 \Rightarrow \frac{1}{\sigma_{11}} \left\{ \sum_1 \left(Z_{11} - \rho Z_{21} \right) + \sum_2 \left(Z_{11} - \rho \frac{Z_{21}}{k_2} \right) + \sum_3 \left(\frac{Z_{11}}{k_1^2} - \frac{\rho Z_{21}}{k_1} \right) + \sum_4 \left(\frac{Z_{11}}{k_1^2} - \frac{\rho Z_{21}}{k_1 k_2} \right) \right\} = 0$$

… ....(4.2)

$$\frac{\partial \log L}{\partial \mu_2} = 0 \Rightarrow \frac{1}{\sigma_{21}} \left\{ \sum_1 \left(Z_{21} - \rho Z_{11} \right) + \sum_2 \left(\frac{Z_{21}}{k_2^2} - \rho \frac{Z_{11}}{k_2} \right) + \sum_3 \left(Z_{21} - \frac{\rho Z_{11}}{k_1} \right) + \sum_4 \left(\frac{Z_{21}}{k_2^2} - \frac{\rho Z_{11}}{k_1 k_2} \right) \right\} = 0$$

....(4.3)

Solving (4.2) and (4.3) for μ_1 and μ_2 we get

$$\hat{\mu}_1 = \frac{k_1^2\left(\sum_1 \log x + \sum_2 \log x\right) + \left(\sum_3 \log x + \sum_4 \log x\right)}{k_1^2\left(r_1 + r_2\right) + \left(r_3 + r_4\right)}$$

…...(4.4)

and

$$\hat{\mu}_2 = \frac{k_2^2\left(\sum_1 \log y + \sum_3 \log y\right) + \left(\sum_2 \log y + \sum_4 \log y\right)}{k_2^2\left(r_1 + r_3\right) + \left(r_2 + r_4\right)}$$

….(4.5)

where r_1, r_2, r_3 and r_4 are the numbers of paired ordered observations satisfied the conditions (i) $\log x \le \mu_1$, $\log y \le \mu_2$, (ii) $\log x \le \mu_1$, $\log y > \mu_2$, (iii) $\log x > \mu_1$, $\log y \le \mu_2$, and (iv) $\log x > \mu_1$, $\log y > \mu_2$ respectively with $\sum_{i=1}^{4} r_i = n$. Further by differentiating (4.1) with respect to σ_{11}, σ_{21} and ρ and equating them to zero and solving them simultaneously we get

$$\hat{\sigma}_{11}^2 = \left[\frac{k_1^2\left\{\sum_1\left(\log x - \mu_1\right)^2 + \sum_2\left(\log x - \mu_1\right)^2\right\} + \left\{\sum_3\left(\log x - \mu_1\right)^2 + \sum_4\left(\log x - \mu_1\right)^2\right\}}{n k_1^2}\right]$$

....(4.6)

$$\hat{\sigma}_{21}^2 = \left[\frac{k^2\left\{\sum_1\left(\log y - \mu_2\right)^2 + \sum_3\left(\log y - \mu_2\right)^2\right\} + \left\{\sum_2\left(\log y - \mu_2\right)^2 + \sum_4\left(\log y - \mu_2\right)^2\right\}}{n k_2^2}\right]$$

....(4.7)

and

$$\hat{\rho} = \frac{k_1 k_2 \sum_1\left(\log x - \mu_1\right)\left(\log y - \mu_2\right) + k_1 \sum_2\left(\log x - \mu_1\right)\left(\log y - \mu_2\right) + k_2 \sum_3\left(\log x - \mu_1\right)\left(\log y - \mu_2\right) + \sum_4\left(\log x - \mu_1\right)\left(\log y - \mu_2\right)}{n k_1 k_2 \ \hat{\sigma}_{11} \ \hat{\sigma}_{21}}$$

….(4.8)

REFERENCES

[1]. Garvin, J. S. and McClean, S.L.(1997). Convolution and sampling theory of the binormal distribution as a prerequisite to its application in statistical process control. The Statistician, 46, No. 1, pp 33-47.

[2]. H.N. Nagaraja (2005). Function of Conconitants of order statistics - ISPS Vol. 7, 16-32.

[3]. John, S. (1982). The three-parameter two-piece normal family of distribution and its fitting. Commun. Statist. – Theor and Meth, 11, 879-885.

[4]. Kimber, A.C. (1985) Methods for the two-piece normal distribution. Commun. Statist. – Theor. And Meth, 14(1), 235-245.

[5]. S. P. Nabar and S. C. Deshmukh. (2002) On Estimation of Parameters of Bilognormal Distribution. Journal of the Indian Statistical Association Vol. 40, 1, 2002, 59-70.

[6]. S.P. Nabar & S.P.Parpande (2001) A note on the maximum likelihood estimator of the binormal distribution BIN (σ, μ, μσ).