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Cone Metric Spaces and Fixed Point Theorems Of T- Contractive Mappings

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ABSTRACT: *In this paper, we obtain sufficient conditions for the existence of a unique fixed point of T- Contraction mapping on complete cone metric spaces.*

KEYWORDS: *Fixed Point, Cone Metric Spaces, Complete Cone Metric Spaces*

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I. INTRODUCTION AND PRELIMINARIES

Huang and **Zhang** [4] generalized the notion of metric spaces, replacing the real numbers by an ordered Banach spaces and define cone metric spaces and also proved some fixed point theorems of contractive type mappings in cone metric spaces.
Now we recall some concept and properties of cone metric spaces.

Definitions and Properties:
In this section we shall define cone metric space and its properties proved by **Huang** and **Zhang** [4].
Let E be a real Banach spaces and $P \subset E$. P is called a cone iff

(i) P is closed and non empty and $P \neq \{0\}$;

(ii) $\forall$ x, y $\in$ P and a, b$\in R^+ \Rightarrow ax + by \in P$
(iii) $P \cap -P = \{0\}$ i.e. $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$ we define a partial ordering $\leq$ with respect to P as

(i) $y - x \in P \Rightarrow x \leq y$;

(ii) $x < y \Rightarrow x \leq y$ but $x \neq y$;

(iii) $x \ll y \Rightarrow y - x \in$ interior of P .(int P)

The cone P is said to be Normal if $\exists$ a number K > 0 such that for all x, y $\in$ E , $0 \leq x \leq y \Rightarrow \|x\| \leq K \|y\|$

The least positive number K satisfying above inequality is called the normal constant of P.

The cone is said to be regular if every increasing sequence which is bounded from above is convergent i.e.

if $\{x_n\}$ is a sequence such $x_1 \leq x_2 \leq x_3 \leq x_4 \ldots\ldots \leq x_n \leq \ldots \leq y$ for some $y \in E$, there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently the cone P is regular iff every decreasing sequence which is bounded from below is convergent. A regular cone is a normal cone.
In the following we always suppose E is Banach space, P is a cone in E with int $P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition: Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies:
(d_1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.
The d is called a cone metric on X, and (X, d) is called a cone metric space.

II. LEMMAS:

3.1 Let (X, d) be cone metric spaces, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if d $(x_n, x) \rightarrow 0$ as $n \rightarrow 0$.

3.2. Let (X, d) be cone metric spaces, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then x = y. The limit of $\{x_n\}$ is unique.

3.3. Let (X, d) be cone metric spaces, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X converging to x, then $\{x_n\}$ is a Cauchy sequence.

3.4. Let (X, d) be cone metric spaces, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ is a Cauchy sequence iff d $(x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \infty$.

3.5. Let (X, d) be cone metric spaces, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \rightarrow x$, $y_n \rightarrow y$ as $n \rightarrow \infty$, then d $(x_n, y_n) \rightarrow d(x, y)$.

III. THEOREMS:

In 2004, **L.G.Haung** and **X. Zhang** [4] proved the following theorem:

Theorem 4.1: Let (X, d) be cone metric spaces, P be a normal cone with normal constant K. Suppose the mapping T: X→ X satisfies the contractive condition

$$d(Tx, Ty) \leq K\, d(x, y) \ \ \textit{for all } x, y \in X,$$

Where k ∈ [0, 1) is a constant. Then T has a unique fixed point in X. And for any x ∈X, iterative sequence

$\{T^n x\}$ converges to a fixed point.

Theorem 4.2: Let (X, d) be cone metric spaces, P be a normal cone with normal constant K. Suppose the mapping T: X→ X satisfies the contractive condition

$$d(Tx, Ty) \leq d(x, y) \ \ \textit{for all } x, y \in X \textit{ and } x \neq y$$

Then T has a unique fixed point in X.

Jose R. Morales and **Edixon Rojas [3]** proved the following theorems:

Theorem 4.3: Let (M, d) be a complete cone metric spaces, P be a normal cone with normal constant K In addition, let:T,S: M→ M be one to one continuous functions and TK_1 - contraction,
(T-Kannan Contraction) Then for all x, y ∈ X.

[i] For every $x_0 \in M$

$$\lim_{n\to\infty} d(TS^n x_0, \ TS^{n+1} x_0) = 0$$

[ii] There is $v \in M$

$$\lim_{n\to\infty} TS^n x_0 = v$$

[iii] If T is sub sequentially convergent, then $(S^n x_0)$ has a convergent subsequence.

[iv] There is a unique u ∈ M such that $Su = u$

[v] If T is sequentially convergent, then for each $x_0 \in M$, iterates sequence $\{S^n x_0\}$ converges to u.

Theorem 4.4: Let (M, d) be a complete cone metric spaces, P be a normal cone with normal constant K
In addition, let S, T: M→ M be one to one continuous functions and, TK_2 – contraction (T-Chatterjea Contraction). Then for all x, y ∈ X.

[i] For every $x_0 \in M$

$$\lim_{n\to\infty} d(TS^n x_0, \ TS^{n+1} x_0) = 0$$

[ii] There is $v \in M$

$$\lim_{n\to\infty} TS^n x_0 = v$$

[iii] If T is sub sequentially convergent, then $(S^n x_0)$ has a convergent subsequence.

[iv] There is a unique u ∈ M such that $Su = u$

[v] If T is sequentially convergent, then for each $x_0 \in M$, the iterates sequence $\{S^n x_0\}$ converges to u.

IV. T-Hardy and Rogers's contraction:

First, we are going to introduce some new definitions on cone metric spaces based on the idea of **Moradi** [7].

5.1. Let (X, d) be a cone metric spaces and T, S: X→ X two functions K_1- A mapping S is said to be T- Hardy and Roger contraction (TK1- contraction) if there is a constant b∈ [0, 1) such that

$$d(TSx, TSy) \leq a_1\, d(Tx, Ty) + a_2\, d(Tx, TSx) + a_3\, d(Ty, TSy) + a_4 d(Tx, TSy) + a_5\, d(Ty, TSx)$$

for all x, y ∈ *X.*

V. MAIN RESULT:

Now we prove the following theorem.

THEOREM 6.1: Let (X, d) be a complete cone metric spaces, P be a normal cone with normal constant K, let in addition T: X→ X be one to one continuous functions and S: X→ X a TK Hardy-Rogers contraction(TK1-Contraction) if there is a constant $b = a_1 + a_2 + a_3 + 2\, a_4 \in [o,1)$ such that

$$d(TSx, TSy) \leq a_1 d(Tx, Ty) + a_2 d(Tx, TSx) + a_3 d(Ty, TSy) + a_4 d(Tx, TSy) + a_5 d(Ty, TSx)$$

for all $x, y \in X$. Then

[1] $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = 0$

[2] $\lim_{n\to\infty} TS^n x_0 = v$

[3] If T is sub sequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence;

[4] If $S_u = u$ then u is unique and $u \in X$ and $\{S^n x_0\}$ converges to u.

Proof: Let $\mathbf{x}_0$ be an arbitrary point in X. A sequence $\{x_n\}$ is defined as $x_{n+1} = Sx_n = S^n x_0$. We have

$$d(Tx_n, Tx_{n-1}) = d(TSx_{n-1}, TSx_n) \leq a_1 d(Tx_{n-1}, Tx_n) + a_2 d(Tx_{n-1}, TSx_{n-1}) + a_3\, d(Tx_n, TSx_n)$$

$$+ a_4\, d(Tx_{n-1}, TSx_n) + a_5 d(Tx_n, TSx_{n-1})$$

$$\leq a_1 d(Tx_{n-1}, Tx_n) + a_2 d(Tx_{n-1}, Tx_n) + a_3\, d(Tx_n, Tx_{n+1}) + a_3\, d(Tx_{n-1}, Tx_{n+1}) + a_3 d(Tx_n, Tx_n)$$

$$(1 - a_3 - a_4)\, d(Tx_n, Tx_{n-1}) \leq (a_1 + a_2 + a_4)\, d(Tx_{n-1}, Tx_n)$$

$$(1-a_3- a_4)\, d(Tx_n, Tx_{n+1}) \leq (a_1 + a_2 + a_4)\, d(Tx_{n-1}, Tx_{n+1})$$

$$d(Tx_n, Tx_{n-1}) \leq \frac{a_1+a_2+a_4}{1-a_3-a_4}\, d(Tx_{n-1}, Tx_{n+1})$$

$$d(Tx_n, Tx_{n-1}) \leq c\, d(Tx_{n-1}, Tx_{n+1}) \qquad (6.1.1)$$

Where $c = \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1 \Longrightarrow a_1 + a_2 + a_3 + 2a_4 < 1$

Repeating the same iteration n times we conclude that

$d(TSx_n, TSx_{n+1}) \leq c^n\ d(Tx_0, TSx_0)$.

i.e. $d(TS^n x_0, TS^{n+1} x_0) \leq c^n d(Tx_0, TSx_0)$ (6.1.2)

$\| d(TSx_n, TSx_{n+1}) \| \leq c^n K \| d(Tx_0, TSx_0) \|$

Or $\| d(TS^n x_0, TS^{n+1} x_0) \| \leq c^n K \| d(Tx_0, TSx_0) \|$

Where, K is the normal constant of E.

By the above inequality we have

$$\lim_{n\to\infty} \| d(TSx_n, TSx_{n+1}) \| = 0$$

Hence $\lim_{n\to\infty} d(TSx_n, TSx_{n+1}) = 0$ *(6.1.3)*

Now from the inequality (1.2), for every m, n $\in N$ with m > n we have

$$d(Tx_n, Tx_m) \leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \ldots\ldots + d(Tx_{m-1}, Tx_m)$$

$$\leq (c^n + c^{n+1} + c^{n+2} + \ldots\ldots + c^{m-1})\ d(Tx_0, TSx_0)$$

$$d(TS_n x_0, TS_m x_0) \leq \frac{c^n}{1-c} d(Tx_0, TSx_0) \quad (6.1.4)$$

From (1.3) we have

$$\| d(TS_n x_0, TS_m x_0) \| \leq \frac{c^n}{1-c}\ K \| d(Tx_0, TSx_0) \|$$

Where K is the normal constant of E. Taking limit m, n→∞ and c < 1 we conclude that

$$\lim_{m, n\to\infty} \| d(TS_n x_0, TS_m x_0) \| = 0$$

Thus $\lim_{m, n\to\infty} d(TS_n x_0, TS_m x_0) = 0$ implies that $\{TS_n x_0\}$ is a Cauchy sequence.

Since X is a complete cone metric space, therefore there exists v $\in$ X such that

$$\lim_{n\to\infty} TS_n x_0 = v \quad (6.1.5)$$

Now if T is sub sequentially convergent, $\{Sx_0\}$ has a convergent subsequence. So there is u$\in$ X and $\{x_{n_i}\}$ such that

$$\lim_{i\to\infty} x_{n_i} = u \quad (6.1.6)$$

Since T is continuous, then

$$\lim_{i\to\infty} TSx_{n_i} = Tu \qquad (6.1.7)$$

Hence from (1.5) and (1.7) we conclude

$$Tu = v \qquad (6.1.8)$$

On the other hand

$d(TSu, Tu) \le dTSu, TS^{n_i}x_0) + (TS^{n_i}x_0, TS^{n_{i+1}}x_0) + d(TS^{n_{i+1}}x_0, Tu)$

$d(TSu, Tu) \le a_1 d(Tu, TSx_{n_{i-1}}) + a_2\, d(Tu, TSu) + a_3 d(TSx_{n_{i-1}}, TSx_{n_i}) + a_4 d(Tu, TSx_{n_i})$

$$+a_5 d(TSx_{n_{i-1}}, TSu) + d(TSx_{n_i}, TSx_{n_{i+1}}) + d(TSx_{n_{i+1}}, Tu)$$

$d(TSu, Tu)$

$$\le \frac{a_1}{1-a_2} d(Tu, TSx_{n_{i-1}}) + \frac{a_3}{1-a_2} d(TSx_{n_{i-1}}, TSx_{n_i}) + \frac{a_1}{1-a_2} d(Tu, TSx_{n_i}) + \frac{c^n}{1-a_2} d(Tx_0, TSx_0)$$

$$\| d(TSu, Tu) \| \le \| \frac{a_1}{1-a_2} K d(Tu, TSx_{n_{i-1}}) \| + \frac{a_3}{1-a_2} K \| d(TSx_{n_{i-1}}, TSx_{n_i}) \|$$

$$+ \frac{a_1}{1-a_2} K \| d(Tu, TSx_{n_i}) \| + \frac{c^n}{1-a_2} K \| d(Tx_0, TSx_0) \| \to 0 \ (i \to 0)$$

Where, K is normal constant of X. This implies that sequence $\{x_n\}$ converges to 0, therefore $d(TSu, Tu) = 0$

$\Longrightarrow TSu = Tu$, but T is one to one mapping, therefore Su = u, consequently S has a fixed point.

If we consider v is another fixed point of S, then from injectivity of T we can easily prove that Su =Sv

As T is sequentially convergent therefore

$\lim_{n\to\infty} S^n x_0 = u. \Longrightarrow \{S^n x_0\}$ converges to the fixed point of S.

This completes the proof of the theorem.

Remarks:

- **(i)** If we take $a_1 = a_4 = a_5 = 0$ *and* $a_2 = a_3 = b$, *we get the theorem* 3.1 *of* [3]
- **(ii)** If we take $a_1 = a_2 = a_3 = 0$ *and* $a_4 = a_5 = c$, *we get the theorem* 3.5 *of* [3]

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