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Formulations and proofs of optimal expressions for control index matrices for a class of double – delay differential equations.

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ABSTRACT: *This paper derived the structure of the indices of control systems for a class of double – delay autonomous linear differential equations on any given interval of length equal to the delay h for non –negative time periods. The formulations and the development of the theorem relied on an earlier work by Ukwu (2013i) on the interval* $[t_1 - 4h, t_1]$*. The derivation of the associated solution matrices exploited the continuity of these matrices for positive time periods, the method of steps and backward continuation recursions to obtain these matrices on successive intervals of length equal to the delay h. The proofs were achieved using ingenious combinations of summation notations, greatest integer functions and multiple integrals. The indices were derived using the stage – wise algorithmic format, starting from the right – most interval of length h. Our results globally extend the time scope of applications of these matrices to the solutions of terminal function problems and rank conditions for controllability and cores of targets.*

KEYWORDS: *Index, Matrices, Recursions, Scalar, Structure.*

I. INTRODUCTION

The importance of indices of control systems matrices stems from the fact that they not only pave the way for the derivation of determining matrices for the determination of Euclidean controllability and compactness of cores of Euclidean targets but can be used independently for such determination. In sharp contrasts to determining matrices the use of indices of control systems for the investigation of the Euclidean controllability of systems can be quite computationally challenging; however this difficulty can be mitigated if the coefficient matrix associated with the state variable at time t is diagonal. This paper pioneers the development of the structure of these indices.

Literature on state space approach to control studies is replete with indices of control systems as key components for the investigation of controllability. See Chukwu (1992), Gabsov and Kirillova (1976), Manitius (1978), Tadmore (1984), and Ukwu (1987, 1992, 1996). Regrettably no author has made any attempt to obtain general expressions for the associated matrices or special cases of such matrices involving the double - delay h and $2h$. Effort is usually focused on the single – delay mode with the usual approach being to start from the interval $[t_1 - h, t_1]$ and compute the index matrices for given problem instances; then the method of steps and backward continuation recursive procedure are deployed to extend these to the intervals $[t_1 - (k+1)h,\ t_1 - kh]$, for positive integral k, not exceeding 2, for the most part. Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary k. In other words such approach fails to address the issue of the structure of control index matrices. The need to address such short-comings has become imperative; this is the major contribution of this paper, in the case the scalar counterparts, with its wide-ranging implications for extensions to systems and holistic approach to controllability studies.

II. PRELIMINARIES

Consider the system:

$$\frac{\partial}{\partial \tau} X(\tau, t) = -X(\tau, t) A_0 - X(\tau + h, t) A_1 - X(\tau + 2h, t) A_2 \tag{1}$$

for $0 < \tau < t, \tau \neq t - kh, k = 0,1,\ldots$ where

$$X(\tau, t) = \begin{cases} I_n; \tau = t \\ 0; \ \tau > t \end{cases} \tag{2}$$

A_0, A_1, A_2 are $n \times n$ constant matrices and $\tau \to X(\tau,t), \tau \to X(\tau, t+h)$ are $n \times n$ matrix functions. See Chukwu (1992), Hale (1977) and Tadmore (1984) for properties of $X(t, \tau)$. Of particular importance is the fact that $\tau \to X(\tau, t)$ is analytic on the intervals $(t_1 - (j+1)h, t_1 - jh)$, $j = 0,1,\ldots;\ t_1 - (j+1)h > 0$. Any such $\tau \in (t_1 - (j+1)h, t_1 - jh)$ is called a regular point of $\tau \to X(t, \tau)$.

2.1 Definition

The expression $c^* X(\tau, t_1) B$ is called the index of a given control system, where c is an n-dimensional constant column vector, $X(\tau, t_1)$ is defined in (1), B is an $n \times m$ constant matrix associated with the control system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) + Bu(t)$$

and $u(.)$ is an m- vector admissible control function. Thus the control index matrix, $X(\tau, t_1)$ determines the structure of the index of a given control system.

We proceed to determine the structure of the above matrix. This will be achieved using the method of steps and a Backward Continuation Recursive procedure.

Let $K_j = \left[t_1 - (j+1)h, t_1 - jh\right], \forall\ j : t_1 - (j+1)h > 0$, and fixed $t_1 > 0$.

Ukwu (2013i) obtained the following expressions for the control index matrices, $X(\tau, t_1)$ on K_j, for $j \in \{0, \cdots, 3\}$:

$$X(\tau, t_1) = \begin{cases} e^{A_0(t_1-\tau)}, \tau \in K_0; & (2) \\ e^{A_0(t_1-\tau)} - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-\tau)} ds_1,\ \tau \in K_1; & (3) \\ e^{A_0(t_1-\tau)} - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-\tau)} ds_1 + \int_{t_1-2h}^{\tau} \int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-h-s_2)} A_1 e^{A_0(s_2-\tau)} ds_1 ds_2 & \\ \quad - \int_{t_1-2h}^{\tau} e^{A_0(t_1-2h-s_3)} A_2 e^{A_0(s_3-\tau)} ds_3, \text{ for } \tau \in K_2; & (4) \end{cases}$$

$$X(\tau, t_1) = \begin{cases} e^{A_0(t_1-\tau)} - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-\tau)} ds_1 + \int_{t_1-2h}^{\tau} \int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-h-s_2)} A_1 e^{A_0(s_2-\tau)} ds_1 ds_2 \\ - \int_{t_1-3h}^{\tau} \int_{t_1-2h}^{s_3+h} \int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-h-s_2)} A_1 e^{A_0(s_2-h-s_3)} A_1 e^{A_0(s_3-\tau)} ds_1 ds_2 ds_3 \\ - \int_{t_1-2h}^{\tau} e^{A_0(t_1-2h-s_3)} A_2 e^{A_0(s_3-\tau)} ds_3 - \int_{t_1-3h}^{\tau} \int_{s_3+2h}^{t_1-h} e^{A_0(s_1-2h-s_3)} A_1 e^{A_0(t_1-h-s_1)} A_2 e^{A_0(s_3-\tau)} ds_1 ds_3 \\ - \int_{t_1-3h}^{\tau} \int_{s_3+h}^{t_1-2h} e^{A_0(s_2-s_3-h)} A_2 e^{A_0(t_1-2h-s_2)} A_1 e^{A_0(s_3-\tau)} ds_2 ds_3, \tau \in K_3 \end{cases} \tag{5}$$

He also interrogated some topological dispositions of the solution matrices and deduced that the solution matrices are continuous on the interval $[t_1 - 4h,\ t_1]$ but not analytic due to the break-down of analyticity for

$\tau \in \{t_1, t_1 - h, t_1 - 2h, t_1 - 3h\}$. These results are consistent with the existing qualitative theory on $X(\tau, t)$. See Chukwu (1992), Hale (1977), Tadmore (1984) and Ukwu (1987, 1996). See also analytic function (2010) and Chidume (2007) for discussions on analytical functions and topology.

The objective of this paper is to formulate and prove a theorem on the general expression for $X(\tau, t_1)$ on K_j, for $j \in \{0, 1, 2, 3\}$, by appropriating the above expression for $X(\tau, t_1)$, for the case $n = 1$, where n is the dimension of the state space.

Let r_0, r_1, r_2 be nonnegative integers and let $P_{0(r_0),1(r_1),2(r_2)}$ denote the set of all permutations of $\underbrace{0,0,\ldots 0}_{r_0 \text{ times}}\ \underbrace{1,1,\ldots 1}_{r_1 \text{ times}}\ \underbrace{2,2,\ldots 2}_{r_2 \text{ times}}$: the permutations of the objects 0, 1, and 2 in which i appears r_i times, $i \in \{0,1,2\}$.

3. Theorem: Ukwu-Garba's Control Index Formula for Autonomous, Double – Delay Linear Systems (1), with state space dimension $n = 1$.

In (1), set $n = 1$, $A_0 = a$, $A_1 = a_1 = b$, $A_2 = a_2 = c$. Then:

$$X(\tau, t_1) = \begin{cases} e^{a(t_1-\tau)}, \ \tau \in K_0; & (6) \\ e^{a(t_1-\tau)} + \sum_{i=1}^{j} b^i \dfrac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1 - [\tau + ih]\right)} \\ + \sum_{k=1}^{\left\lceil \left\lfloor \frac{j}{2} \right\rfloor \right\rceil} \sum_{i=0}^{j-2k} b^i c^k \dfrac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{i!k!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)}, \ \tau \in K_j, \ j \geq 1 & (7) \end{cases}$$

The third component:

$$\sum_{k=1}^{\left\lceil \left\lfloor \frac{j}{2} \right\rfloor \right\rceil} \sum_{i=0}^{j-2k} b^i c^k \frac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{i!k!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)}$$
$$= \sum_{k=1}^{\left\lceil \left\lfloor \frac{j}{2} \right\rfloor \right\rceil} \sum_{i=0}^{j-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i),2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{(i+k)!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)} \quad (8)$$

The equivalent form in (8) will be exploited to achieve the proof of the theorem.

Note that the formula can be rewritten in the form:

$$X(\tau, t_1) = I_n e^{a(t_1-\tau)} + \sum_{i=1}^{j} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1 - [\tau + ih]\right)} \operatorname{sgn}(\max\{0, j\})$$
$$+ \sum_{k=1}^{\left\lceil \left\lfloor \frac{j}{2} \right\rfloor \right\rceil} \sum_{i=0}^{j-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i),2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{(i+k)!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)} \operatorname{sgn}(\max\{0, j-1\}) \quad (9)$$

Proof

First, we prove that the theorem is true for $\tau \in K_j$, $j \in \{0, 1, 2, 3\}$ by comparing the results with expressions (2) through (5) above. Then we use induction to complete the proof:

$$t \in K_0 \Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)};\ (2) \Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)};$$

$$t \in K_1 \Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} - b(\tau + h - t_1)e^{-a(\tau+h-t_1)} = e^{a(t_1-\tau)} + b(t_1 - [\tau + h])e^{a(t_1-[\tau+h])}$$

$$(3) \Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} - \int_{t_1-h}^{\tau} b e^{a(t_1-[\tau+h])} ds = e^{a(t_1-\tau)} + b(t_1 - [\tau + h])e^{a(t_1-[\tau+h])}$$

$$\tau \in K_2 \Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + \sum_{i=1}^{2} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)} + \sum_{v_1 \in P_{1(0),2(1)}} a_{v_1} \frac{\left(t_1 - [\tau + (0+2)h]\right)^{0+1}}{(0+1)!} e^{a\left(t_1-[\tau+(0+2)h]\right)}$$

$$\Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + \sum_{i=1}^{2} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)} + c\left(t_1 - [\tau + 2h]\right)e^{a\left(t_1-[\tau+2h]\right)}$$

$$(4) \Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + b\left(t_1 - [\tau + h]\right)e^{a\left(t_1-[\tau+h]\right)} + \int_{t_1-2h}^{\tau} \int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} b e^{A_0(s_1-h-s_2)} b e^{A_0(s_2-\tau)} ds_1 ds_2 - \int_{t_1-2h}^{\tau} e^{a(t_1-2h-s_3)} c e^{a(s_3-\tau)} ds_3$$

$$\Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + b\left(t_1 - [\tau + h]\right)e^{a\left(t_1-[\tau+h]\right)} + c\left(t_1 - [\tau + 2h]\right)e^{\left(t_1-[\tau+2h]\right)} + b^2 \frac{\left(t_1 - [\tau + h]\right)^2}{2} e^{a\left(t_1-[\tau+2h]\right)}$$

$$\Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + \sum_{i=1}^{2} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)} + c\left(t_1 - [\tau + 2h]\right)e^{\left(t_1-[\tau+2h]\right)}$$

$$\tau \in K_3 \Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + \sum_{i=1}^{j} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)} + \sum_{i=0}^{1} \sum_{(v_1, v_2, \cdots, v_{i+1}) \in P_{1(i),2(1)}} a_{v_1} a_{v_2} \cdots a_{v_{i+1}} \frac{\left(t_1 - [\tau + (i+2)h]\right)^{i+1}}{(i+1)!} e^{a\left(t_1-[\tau+(i+2)h]\right)}$$

$$\Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + \sum_{i=1}^{j} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)} + c\left(t_1 - [\tau + 2h]\right)e^{a\left(t_1-[\tau+2h]\right)} + \left(bc + cb\right)\frac{\left(t_1 - [\tau + 3h]\right)^2}{2!} e^{a\left(t_1-[\tau+3h]\right)}$$

$$(5) \Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + \sum_{i=1}^{2} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)}$$

$$+ c\left(t_1 - [\tau + 2h]\right)e^{\left(t_1-[\tau+2h]\right)} - \int_{t_1-3h}^{\tau} \int_{t_1-2h}^{s_3+h} \int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} b e^{a(s_1-h-s_2)} b e^{a(s_2-h-s_3)} b e^{a(s_3-\tau)} ds_1 ds_2 ds_3$$

$$- \int_{t_1-2h}^{\tau} e^{a(t_1-2h-s_3)} c\, e^{a(s_3-\tau)} ds_3 - \int_{t_1-3h}^{\tau} \int_{s_3+2h}^{t_1-h} e^{a(s_1-2h-s_3)} b e^{a(t_1-h-s_1)} c e^{a(s_3-\tau)} ds_1 ds_3$$

$$- \int_{t_1-3h}^{\tau} \int_{s_3+h}^{t_1-2h} e^{a(s_2-s_3-h)} c\, e^{a(t_1-2h-s_2)} b e^{A_0(s_3-\tau)} ds_2 ds_3$$

$$\Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + \sum_{i=1}^{3} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)} + c\left(t_1 - [\tau + 2h]\right) e^{\left(t_1-[\tau+2h]\right)}$$
$$+ \left(bc + cb\right) \frac{\left(t_1 - [\tau + 3h]\right)^2}{2!} e^{a\left(t_1-[\tau+3h]\right)},$$

needless to say: $\left(bc + cb\right) \frac{\left(t_1 - [\tau + 3h]\right)^2}{2!} e^{a\left(t_1-[\tau+3h]\right)} = bc\left(t_1 - [\tau + 3h]\right)^2 e^{a\left(t_1-[\tau+3h]\right)}$

Therefore the theorem has been be verified for $j \in \{0,1,2,3\}$. Assume that the theorem is valid for $\tau \in K_p, 4 \le p \le j$, for some integers p and j. Then $\tau, s_{j+1} \in K_{j+1} \Rightarrow t_1 - [j+1]h \in K_j, \ \ s_{j+1} + h \in K_j$ and $s_{j+1} + 2h \in K_{j-1}$. Hence:

$$\Rightarrow X(\tau, t_1) = X(t_1 - [j+1]h, t_1) e^{a(t_1-[j+1]h-\tau)} - \int_{t_1-[j+1]h}^{\tau} X(s_{j+1} + h, t_1) b e^{a(s_{j+1}-\tau)} ds_j$$

$$- \int_{t_1-[j+1]h}^{\tau} X(s_{j+1} + 2h, t_1) c e^{a(s_{j+1}-\tau)} ds_{j+1} \quad (10)$$

$$= e^{a\left(t_1-\tau\right)} + \sum_{i=1}^{j} \frac{b^i \left([j+1-i]h\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)} \quad (11)$$

$$+ \sum_{k=1}^{\left\lfloor\left\lceil \frac{j}{2} \right\rceil\right\rfloor} \sum_{i=0}^{j-2k} \sum_{(v_1,v_2,\cdots,v_{i+k})\in P_{1(i),2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left([j+1-i-2k]h\right)^{i+k}}{(i+k)!} \quad (12)$$

$$+ b\left(t_1 - [\tau + (j+1)h]\right) e^{a(t_1-\tau)} - \int_{t_1-(j+1)h}^{t} \sum_{i=1}^{j} b^{i+1} \frac{\left(t_1 - [s_{j+1} + (i+1)h]\right)^i}{i!} e^{a\left(t_1-[\tau+(i+1)h]\right)} ds_{j+1} \quad (13)$$

$$- \int_{t_1-(j+1)h}^{\tau} \sum_{k=1}^{\left\lfloor\left\lceil \frac{j}{2} \right\rceil\right\rfloor} \sum_{i=0}^{j-2k} \sum_{(v_1,v_2,\cdots,v_{i+k})\in P_{1(i),2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left(t_1 - [s_{j+1} + (i+1+2k)h]\right)^{i+k}}{(i+k)!} b e^{a\left(t_1-[\tau+(i+1+2k)h]\right)} ds_{j+1} \quad (14)$$

$$+ c\left(t_1 - [\tau + (j+1)h]\right) e^{a(t_1-\tau)} - \int_{t_1-(j+1)h}^{\tau} \sum_{i=1}^{j} b^i c \frac{\left(t_1 - [s_{j+1} + (i+2)h]\right)^i}{i!} e^{a\left(t_1 - [\tau + (i+2)h]\right)} ds_{j+1} \qquad (15)$$

$$- \int_{t_1-(j+1)h}^{\tau} \sum_{k=1}^{\left\lceil\left\lfloor \frac{j-1}{2} \right\rfloor\right\rceil} \sum_{i=0}^{j-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i),2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left(t_1 - [s_{j+1} + (i+2+2k)h]\right)^{i+k}}{(i+k)!} c e^{a\left(t_1 - [\tau + (i+2+2k)h]\right)} ds_{j+1} \quad (16)$$

The expression (13) yields:

$$b\left(t_1 - [\tau + (j+1)h]\right) e^{a(t_1-\tau)} + \sum_{i=2}^{j+1} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1 - [\tau + ih]\right)} - \sum_{i=2}^{j+1} \frac{b^i \left([j+1-i]h\right)^i}{i!} e^{a\left(t_1 - [\tau + ih]\right)} \quad (17)$$

The expression (14) yields:

$$\sum_{k=1}^{\left\lceil\left\lfloor \frac{j}{2} \right\rfloor\right\rceil} \sum_{i=0}^{j-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i-1),2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{(i+k)!} b e^{a\left(t_1 - [\tau + (i+2k)h]\right)} \qquad (18)$$

$$- \sum_{k=1}^{\left\lceil\left\lfloor \frac{j}{2} \right\rfloor\right\rceil} \sum_{i=0}^{j-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i-1),2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left([j+1-i-2k]h\right)^{i+k}}{(i+k)!} b e^{a\left(t_1 - [\tau + (i+2k)h]\right)} \qquad (19)$$

since the summations with $i = 0$ are infeasible and so may be equated to zero, yielding:

$$\sum_{k=1}^{\left\lceil\left\lfloor \frac{j}{2} \right\rfloor\right\rceil} \sum_{i=0}^{j-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i),2(k)}} A_{v_1} A_{v_2} \cdots A_{v_{i+k}} \frac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{(i+k)!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)}, \qquad (20)$$

(with a trailing $a_2 = c$)

$$- \sum_{k=1}^{\left\lceil\left\lfloor \frac{j}{2} \right\rfloor\right\rceil} \sum_{i=0}^{j-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i),2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left([j+1-i-2k]h\right)^{i+k}}{(i+k)!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)}, \qquad (21)$$

with a trailing $a_2 = c$

The expression (15) yields:

$$c\left(t_1 - [\tau + (j+1)h]\right) e^{a(t_1-\tau)} + \sum_{i=1}^{j-1} b^i c \frac{\left(t_1 - [\tau + (i+2)h]\right)^{i+1}}{(i+1)!} e^{a\left(t_1 - [\tau + (i+2)h]\right)}$$

$$- \sum_{i=1}^{j-1} \frac{b^i c \left([j-1-i]h\right)^{i+1}}{(i+1)!} e^{a\left(t_1 - [\tau + (i+2)h]\right)} \qquad (22)$$

The expression (16) yields:

$$\sum_{k=2}^{\left\lceil\left\lfloor \frac{j+1}{2} \right\rfloor\right\rceil} \sum_{i=0}^{j+1-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i),2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{(i+k)!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)} \qquad (23)$$

(with a trailing $a_2 = c$)

$$-\sum_{k=2}^{\left\lfloor\left\lceil\frac{j+1}{2}\right\rceil\right\rfloor}\sum_{i=0}^{j+1-2k}\sum_{(v_1,v_2,\cdots,v_{i+k})\in P_{1(i),2(k)}} a_{v_1}a_{v_2}\cdots a_{v_{i+k}}\frac{\left([j+1-i-2k]h\right)^{i+k}}{(i+k)!}e^{a\left(t_1-[\tau+(i+2k)h]\right)} \quad (24)$$

(with a trailing $a_2 = c$)

$$=\sum_{k=1}^{\left\lfloor\left\lceil\frac{j+1}{2}\right\rceil\right\rfloor}\sum_{i=0}^{j+1-2k}\sum_{(v_1,v_2,\cdots,v_{i+k})\in P_{1(i),2(k)}} a_{v_1}a_{v_2}\cdots a_{v_{i+k}}\frac{\left(t_1-[\tau+(i+2k)h]\right)^{i+k}}{(i+k)!}e^{a\left(t_1-[\tau+(i+2k)h]\right)} \quad (25)$$

(with a trailing $a_2 = c$)

$$-\sum_{i=0}^{j-1} b^i c\,\frac{\left(t_1-[\tau+(i+2)h]\right)^{i+1}}{(i+1)!}e^{a\left(t_1-[\tau+(i+2)h]\right)} \quad (26)$$

$$-\sum_{k=1}^{\left\lfloor\left\lceil\frac{j+1}{2}\right\rceil\right\rfloor}\sum_{i=0}^{j+1-2k}\sum_{(v_1,v_2,\cdots,v_{i+k})\in P_{1(i),\,2(k)}} a_{v_1}a_{v_2}\cdots a_{v_{i+k}}\frac{\left([j+1-i-2k]h\right)^{i+k}}{(i+k)!}e^{a\left(t_1-[\tau+(i+2k)h]\right)} \quad (27)$$

(with a trailing $a_2 = c$)

$$+\sum_{i=0}^{j-1} b^i c\,\frac{\left([j-1-i]h\right)^{i+1}}{(i+1)!}e^{a\left(t_1-[\tau+(i+2k)h]\right)} \quad (28)$$

Therefore:

$Y(t) = (11)+(12)+(17)+(20)+(21)+(22)+(25)+(26)+(27)+(28) = (29)+(30)+\cdots+(38)$

$$= e^{a\left(t_1-\tau\right)}+\sum_{i=1}^{j}\frac{b^i\left([j+1-i]h\right)^i}{i!}e^{a\left(t_1-\left[\tau+ih\right]\right)} \quad (29)$$

$$+\sum_{j=1}^{\left\lfloor\left\lceil\frac{j+1}{2}\right\rceil\right\rfloor}\sum_{i=0}^{j+1-2k}\sum_{(v_1,v_2,\cdots,v_{i+k})\in P_{1(i),\,2(k)}} a_{v_1}a_{v_2}\cdots a_{v_{i+j}}\frac{\left([j+1-i-2k]h\right)^{i+k}}{(i+k)!}e^{a\left(t_1-[\tau+(i+2k)h]\right)} \quad (30)$$

$$b\left(t_1-\left[\tau+(j+1)h\right]\right)e^{a(t_1-\tau)}+\sum_{i=2}^{j+1} b^i\frac{\left(t_1-[\tau+ih]\right)^i}{i!}e^{a\left(t_1-[\tau+ih]\right)}-\sum_{i=2}^{j+1}\frac{b^i\left([j+1-i]h\right)^i}{i!}e^{a\left(t_1-[\tau+ih]\right)} \quad (31)$$

$$+\sum_{k=1}^{\left\lfloor\left\lceil\frac{j+1}{2}\right\rceil\right\rfloor}\sum_{i=0}^{j+1-2k}\sum_{(v_1,v_2,\cdots,v_{i+k})\in P_{1(i),\,2(k)}} a_{v_1}a_{v_2}\cdots a_{v_{i+k}}\frac{\left(t_1-[\tau+(i+2k)h]\right)^{i+k}}{(i+k)!}e^{a\left(t_1-[\tau+(i+2k)h]\right)} \quad (32)$$

(with a leading $a_1 = b$)

$$\left(\text{noting that } k \text{ even} \Rightarrow \left\lfloor\left\lceil\frac{j}{2}\right\rceil\right\rfloor=\left\lceil\left\lceil\frac{j}{2}\right\rceil\right\rceil;\ j \text{ odd } \Rightarrow \left\lceil\left\lfloor\frac{j}{2}\right\rfloor\right\rceil=\left\lceil\left\lfloor\frac{j+1}{2}\right\rfloor\right\rceil-1 \text{ and the fact that}\right)$$

$$\left(j \text{ odd}, k = \left\lceil\left\lfloor \frac{j+1}{2} \right\rfloor\right\rceil \Rightarrow j+1-2k = 0 \Rightarrow \sum_{i=0}^{j+1-2k} (.) = 0, \text{being infeasible. So } \sum_{k=1}^{\left\lceil\left\lfloor \frac{j+1}{2} \right\rfloor\right\rceil} (.) \text{ is appropriate.} \right)$$

$$- \sum_{k=1}^{\left\lceil\left\lfloor \frac{j+1}{2} \right\rfloor\right\rceil} \sum_{i=0}^{j+1-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i),\, 2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left([j+1-i-2k]h\right)^{i+k}}{(i+k)!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)} \qquad (33)$$

$\left(\text{with a leading } a_1 = b\right)$

$$+ c\left(t_1 - [\tau + (j+1)h]\right) e^{a(t_1 - \tau)} + \sum_{i=1}^{j-1} b^i c \frac{\left(t_1 - [\tau + (i+2)h]\right)^{i+1}}{(i+1)!} e^{a\left(t_1 - [\tau + (i+2)h]\right)}$$

$$- \sum_{i=1}^{j-1} \frac{b^i c\left([j-1-i]h\right)^{i+1}}{(i+1)!} e^{a\left(t_1 - [\tau + (i+2)h]\right)} \qquad (34)$$

$$+ \sum_{k=1}^{\left\lceil\left\lfloor \frac{j+1}{2} \right\rfloor\right\rceil} \sum_{i=0}^{j+1-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i),\, 2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left(t - [i+2j]h\right)^{i+k}}{(i+k)!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)} \qquad (35)$$

$(\text{with a leading } a_2 = c)$

$$- \sum_{i=0}^{j-1} b^i c \frac{\left(t - [i+2]h\right)^{i+1}}{(i+1)!} e^{a\left(t_1 - [\tau + (i+2)h]\right)} \qquad (36)$$

$$- \sum_{j=1}^{\left\lceil\left\lfloor \frac{j+1}{2} \right\rfloor\right\rceil} \sum_{i=0}^{j+1-2k} \sum_{(v_1, v_2, \cdots, v_{i+k}) \in P_{1(i),\, 2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left([j+1-i-2k]h\right)^{i+k}}{(i+k)!} e^{a\left(t_1 - [\tau + (i+2k)h]\right)} \qquad (37)$$

$(\text{with a leading } a_2 = c)$

$$+ \sum_{i=0}^{j-1} b^i c \frac{\left([j-1-i]h\right)^{i+1}}{(i+1)!} e^{a\left(t_1 - [\tau + (i+2)h]\right)} \qquad (38)$$

(29)+(31) yields:

$$e^{a(t_1-\tau)} + \sum_{i=1}^{j+1} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)} \qquad (39)$$

(32) + (35)

yields: $$\sum_{k=1}^{\left\lfloor\left\lceil \frac{j+1}{2} \right\rceil\right\rfloor} \sum_{i=0}^{j+1-2k} \sum_{(v_1,v_2,\cdots,v_{i+k})\in P_{1(i),\,2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{(i+k)!} e^{a\left(t_1-[\tau+(i+2k)h]\right)} \qquad (40)$$

Expressions (30) + (33) + (37) yield zero; the expressions cancel out.
Expressions (34) + (34) + (38) yield zero; the expressions cancel out.
Therefore, on K_{j+1}, $X(\tau, t_1)$ reduces to $X(\tau, t_1) =$ expression (39) + expression (40)

$$\Rightarrow X(\tau, t_1) = e^{a(t_1-\tau)} + \sum_{i=1}^{j+1} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)}$$

$$+ \sum_{k=1}^{\left\lfloor\left\lceil \frac{j+1}{2} \right\rceil\right\rfloor} \sum_{i=0}^{j+1-2k} \sum_{(v_1,v_2,\cdots,v_{i+k})\in P_{1(i),\,2(k)}} a_{v_1} a_{v_2} \cdots a_{v_{i+k}} \frac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{(i+k)!} e^{a\left(t_1-[\tau+(i+2k)h]\right)} \qquad (41)$$

for $\tau \in K_{j+1}$. This completes the proof of the theorem.

3.1 Corollary

If $A_0 = a = 0$, then

$$X(\tau, t_1) = \begin{cases} 1, \ \tau \in K_0; & (42) \\ 1 + \sum_{i=1}^{j} b^i \frac{\left(t_1 - [\tau + ih]\right)^i}{i!} \\ + \sum_{k=1}^{\left\lfloor\left\lceil \frac{j}{2} \right\rceil\right\rfloor} \sum_{i=0}^{j-2k} b^i c^k \frac{\left(t_1 - [\tau + (i+2k)h]\right)^{i+k}}{i!\,k!}, \ \tau \in K_j, \ j \geq 1 & (43) \end{cases}$$

Proof

The proof is immediate, noting that $A_0 = 0 \Rightarrow e^{a(.)} = 1$.

III. CONCLUSION

This paper relied greatly on the optimal deployment of combinatorial analysis and change of variables technique, without which its development would be impossible. The paper has provided a sound basis for its extension to more general systems. Such extension must of necessity rely on multinomial distributions, the method of steps and backward continuation recursive procedure.

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