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A Note on the Unknotting Number of Welded Knots

Zhiguo Li[1]Fengchun Lei[2] Jie Wu[3]
[1]*Department of Mathematics,Hebei University of Techonology, Tianjin-300401,P.R.China*
[2]*Department of Mathematics, Dalian University of Technology, Dalian-116024,P.R.China*
[3]*Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing 101408,P.R.China,*

ABSTRACT: *The unknotting numberof a knot is the smallest number of crossing changes required to obtain the unknot, the minimum taken over all theregular projections of this knot. In this paper, we extend a lower bound of the unknotting number of classical knots to welded knots.*

KEYWORDS: *Welded knot, Unknotting number, virtual knots*

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I. INTRODUCTION

The virtual knot theory in [4] and welded knot in theory [2] are two generalizationsof classical knot theory in Euclidian three space. The basic task to study theknot theory is to find knot invariants. Many knot and link invariants have beendiscovered and studied, mostly in the 20th and 21st centuries. Several classicalknot invariants can be extended to those of virtual or welded knots. For example,the knot group and the knot quandle are invariants of virtual knots and weldedknots. The Jones polynomial is a well-defined invariant of virtual links [4] but notwell defined to welded links. Hence there is some differences between these two theories.It will be interesting to study the difference between them.

A virtual knot diagram is a generic immersion of the circle into the plane, withdouble points representing classical crossings and virtual crossings. The branchesof a classical crossing are divided into an overpass and an underpass in a usual way.A virtual crossing is represented by two crossing arcs with a small circle placedaround the crossing point.Two virtual knot diagrams are equivalent under ambient isotopy and some typesof local moves (generalized Reidemeister moves): Classical Reidemeister moves (R1-R3), virtual Reidemeister moves (V1-V3) and mixed Reidemeister moves (V4) (seeFigure 1).

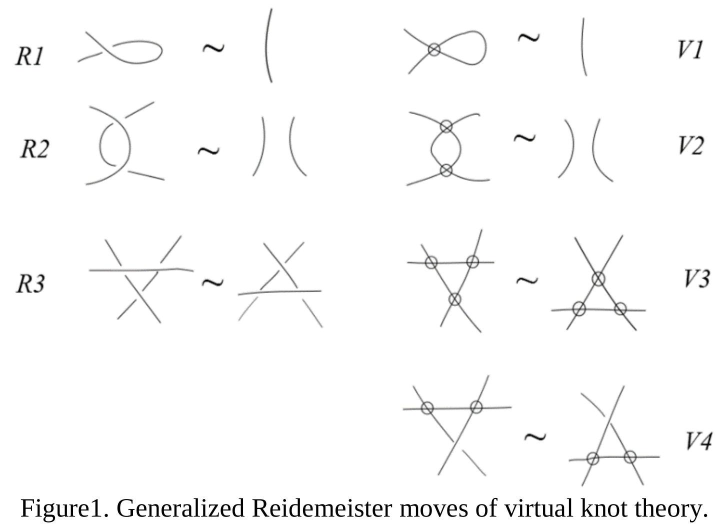

Figure1. Generalized Reidemeister moves of virtual knot theory.

A virtual knot is the equivalence class of a virtual knot diagram. To the generalized Reidemeister moves on virtual diagrams one could add the following localmoves in Figure 2, called forbidden moves of type F1 (left) and F2 (right).If allowed the forbidden move F1, then one can obtain the theory of Weldedlinks whose interest is growing up recently because of the fact that the weldedbraid counterpart can be defined in several equivalent ways (for instance in termsof configuration spaces, mapping classes and automorphisms of free groups [1]). Ifallowed two forbidden moves, one can get the theory called Fused knot or Unweldedknot which is a trivial theory, since any knot in this theory is equivalent to thetrivial knot [3, 6].

Figure 2. Forbidden moves F1(left), F2(right).

Several invariants in the classical knot theory can be naturally extended to avirtual knot, including the Jones polynomial and the knot group $G(K)$. As thesame as the Wirtinger presentation for a classical knot, one can apply the sameprocedure to associate to a virtual knot K an abstract group $G(K)$. The group $G(K)$ has one generator for each arc starting from an under-crossing and endingat an under-crossing, and one relation for each classical crossing, just as in Figure3. In this construction, we ignore the virtual crossing of K. This construction wasfirst introduced by Kauffman in [4]. The knot group $G(K)$ is invariant under allgeneralized Reidemeister moves and hence is an invariant of a virtual knot. Theknot group $G(K)$ does not also change under the forbidden move F1. So the knotgroup $G(K)$ is an invariant of a welded knot. It is still an open problem whether $G(K)$ detects the trivial knot among welded knots.

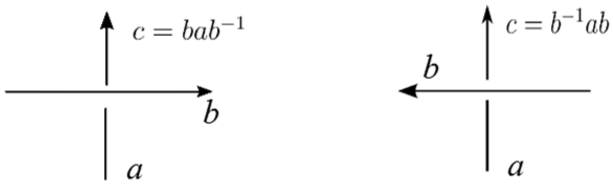

Figure3. Relation at a crossing.

Let K be an oriented tame knot in a 3-sphere S^3, and let K be a diagram i.e.the image under a regular projection of S^3 into S^2. On all diagrams representing K, the minimum number of exchanges of overcrossings and undercrossings requiredto deform K into a trivial knot is called the unknotting nunber of K, denoted by $u(K)$, and the minimum number of crossings is called the crossing number of K ,denoted by $C(K)$.Let G be the fundamental group of $S^3 - N(K)$, and G' be the commutatorsubgroup of G. Then G' is the normal subgroup of G such that

$$G/G' = Z.$$

The group G is the semi-product of Z and G', that is, there is a homomorphism from Z to $Aut(G')$ as in [2]. If there are n elements, say $x_1, x_2, ..., x_n$, in G' such that G' isthe normal closure of $x_1, x_2, ..., x_n$ in G, that is, $G' = < x_1, x_2, ..., x_n >^G$,and if n is minimal along all such presentations of G' in G, then we define $a(K) = n.$ Since K is trivial in S^3 iff $\pi_1\left(S^3 - N\left(K\right)\right)$ is infinite cyclic, $a(K) = 0$ iff K is trivial in S^3.

In 2006, Ma and Qiu[5] showed that $u(K) \geq a(K)$ for any classical knot K. In thisnote, we generalize this result to welded knot theory.

Theorem 1. For a welded knot K, $u(K) \geq a(K)$.

II. THEPROOF OF THEOREM 1

Simillar to the methods of Ma and Qiu, we give a modified Wirtinger presentation asfollows.Let K be a welded knot with a given orientation and $D(K)$ be a projectiondiagram of K with n classical crossings, say $v_1, v_2, ..., v_n$. Now the n crossings separate $D(K)$ into $2n$ arcs, we denote the $2n$ arcs by $a_1, a_2, ..., a_{2n}$ such that a_i connectswith a_{i-1} and a_{i+1} (mod2n) as in Figure 4(left). Then we obtain a presentation of $G(K)$ such that each arc aiinduces a generator, denoted by x_i, and each crossing v_j induces two relations C_j and A_j, where C_j is $x_{k+1}x_{i+1} = x_i x_k$, and A_j is $x_k = x_{k+1}$ for some i and k as in Figure 4(right).Comparing with the Wirtinger presentation, note that in Wirtinger presentation,n crossings separate $D(K)$ into just n arcs, and C_j is the proper relation given bythe crossing in the Wirtinger presentation, and A_j is just the relation that identifies x_k and x_{k+1}. Since they are the same generator in the Wirtinger presentation. Sothis modified Wirtinger presentation gives a presentation of $G(K)$ such that

$$G(K) = \langle x_1, x_2, ..., x_{2n} \mid C_1, C_2, ..., C_n, A_1, A_2, ..., A_n \rangle.$$

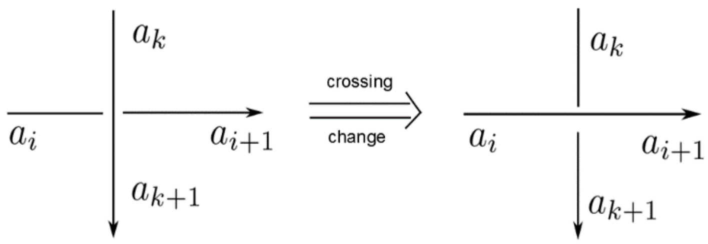

Figure4. Labeling under the crossing change.

Proof of Theorem 1. Let B_j be the relation $x_i = x_{i+1}$ for the crossing v_j inwhich C_j is $x_{k+1}x_{i+1} = x_i x_k$, and A_j is $x_k = x_{k+1}$ for some i and k. Then, after doing a crossingchange on v_j, A_j transforms to B_j, but C_j is the same, as in Figure 2. Since, afterdoing $u(D(K)) = u$ times crossing changes, we obtain a trivial knot, so

$$\langle x_1, x_2, ..., x_{2n} \mid C_1, C_2, ..., C_n, B_{j_1}, B_{j_2}, ..., B_{j_u}, A_{j_{u+1}}, A_{j_{u+2}}, ..., A_{j_n} \rangle$$

is an infinite cyclic group, where $\{j_1, j_2, ..., j_n\} = \{1, 2, ..., n\}$.

Since $[x_i]$ is the generator of $G/G' = Z$,so

$$\langle x_1, x_2, ..., x_{2n} \mid C_1, C_2, ..., C_n, B_{j_1}, B_{j_2}, ..., B_{j_u}, A_{j_1}, A_{j_2}, ..., A_{j_n} \rangle$$

is also the infinite cyclic group. Since $B_{j_1}, B_{j_2}, ..., B_{j_u} \in G', G' = \langle B_{j_1}, B_{j_2}, ..., B_{j_u} \rangle^G$,hence

$$u(D(K)) \geq a(K) \text{ for any } D(K).$$

That means $u(K) \geq a(K)$.

III. CONCLUSION

The smallest number of times of crossing changes of a knot must be passed through itself to untie it. Lower bounds can be computed using relatively straightforward techniques, but it is in general difficult to determine exact values.There are many algebraic invariants of classical knots which have relations with unknotting number. It is worth extending these invariants to virtual knot theory and welded knot theory.

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