International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 11 Issue 6 || Nov. – Dec., 2023 || PP 15-20

The general atom-bond connectivity index for some graphs

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ABSTRACT: *The atom-bond connectivity index plays a key role in correlating the physical-chemical properties and molecular structures of some families of compound. The general atom-bond connectivity index is a generalization of the atom-bond connectivity index. In this paper, we obtain some bounds of the general atom-bond connectivity index for connected graphs with given clique number and trees with given pendant number, and characterize the corresponding extremal graphs.*
KEYWORDS: *The general atom-bond connectivity index, Clique number, Pendant number and Trees.*
AMS Subject Classification(2010): *05C35*

Date of Submission: 20-12-2023 Date of acceptance: 31-12-2023

I. Introduction

The atom-bond connectivity index plays a key role in correlating the physical-chemical properties and molecular structures of some families of compound. The heat of formation in alkanes is predicted or reproduced by [7, 10]. The differences in the energy of linear and branched alkanes both qualitatively and quantitatively are explained by [6]. The extremal values of the atom-bond connectivity index among graphs under various constrains have been extensively explored by [1,3,5,9,11,12,13,15]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $d(v)$ be the degree of $v \in V(G)$. [8] Considered the following generalization :

$$ABC_\alpha(G) = \sum_{uv\in E(G)} \left(\frac{d(u)+d(v)-2}{d(u)d(v)}\right)^\alpha$$

for any $\alpha \in R \setminus \{0\}$, and called it the general atom-bond connectivity index. The optimization problems for the general atom-bond connectivity index have been and are being studied recently, see [2,4,8,16]. Characterizing such graphs with maximum and minimum general atom-bond connectivity index is an interesting work. This motivates our research on the general atom-bond connectivity index for connected graphs with given clique number and trees with given pendant number.

II. Results for connected graphs with given clique number

Let $N(v)$ be the set of neighbors of $v \in V(G)$. Denote by Δ and δ the maximum and minimum vertex degree in G respectively. Denote by K_n the complete graphs of order n. The number of vertices of the largest clique in a graph is called its clique number. For a positive integer q, a graph is called balanced complete q-partite graph if it is a complete q-partite in which all classes are of equal cardinality.

In order to prove our result, the following lemmas are needed.

Lemma 1([14]**)** Let G be a connected K_{q+1}-free graph of order n and size m. Then

$$m \le \left(1-\frac{1}{q}\right)\frac{n^2}{2}$$

with equality iff G is a balanced complete q-partite graph.

Lemma 2 Let $f(x,y)=\left(\frac{x+y-2}{xy}\right)^\alpha$, where $x, y \ge 1$ and $\alpha \in \mathrm{R} \setminus \{0\}$. Then

(i) If $\alpha < 0$, then $f(1,y)$ is decreasing on $[2,+\infty)$. If $\alpha > 0$, then $f(1,y)$ is increasing on $[2,+\infty)$.

(ii) $f(2,y) = \left(\frac{1}{2}\right)^\alpha$ for every $y \ge 1$.

(iii) If $\alpha < 0$, then $f(x,y)$ is increasing in each variable on $[2,+\infty)$. If $\alpha > 0$, then $f(x,y)$ is decreasing in each variable on $[2,+\infty)$.

Proof. (ii) is direct. Note that $f(1,y) = \left(1-\frac{1}{y}\right)^\alpha$ and $1-\frac{1}{y}$ is increasing for $y \ge 2$. Thus (i) holds.

Recall that $f(x,y)=\left(\frac{x+y-2}{xy}\right)^{\alpha}=\left[\frac{1}{x}+\frac{1}{y}\left(1-\frac{2}{x}\right)\right]^{\alpha}$. Also note that $1-\frac{2}{x}\geq 0$ for $x\geq 2$ and $\frac{1}{x}+\frac{1}{y}\left(1-\frac{2}{x}\right)$ is decreasing for $y\geq 2$. If $\alpha<0$, then $f(x,y)$ is increasing for $y\geq 2$. If $\alpha>0$, then $f(x,y)$ is decreasing for $y\geq 2$. By symmetry, the case of x also holds. Thus (iii) holds. ■

Now we give an upper bound on the general atom-bond connectivity index for connected graphs with given clique number.

Theorem 3 Let G be a connected graph of order n with clique number q. If $\alpha<0$, then

$$ABC_{\alpha}(G)\leq \frac{n^2(q-1)}{2q}\left[\frac{2(\Delta-1)}{\Delta^2}\right]^{\alpha};$$

If $\alpha>0$, then for $\delta\geq 2$,

$$ABC_{\alpha}(G)\leq \Delta\left(\frac{\Delta+\delta-2}{\Delta\delta}\right)^{\alpha}+\left[\frac{(q-1)n^2}{2q}-\Delta\right]\left[\frac{2(\delta-1)}{\delta^2}\right]^{\alpha}$$

with equalities iff G is a balanced complete q-partite graph.

Proof. If $\alpha<0$, then for any $v_iv_j\in E(G)$, by Lemma 2 (iii),

$$f\left(d(v_i),d(v_j)\right)\leq f(\Delta,\Delta)=\left[\frac{2(\Delta-1)}{\Delta^2}\right]^{\alpha}$$

with equality iff $d(v_i)=d(v_j)=\Delta$. Thus

$$\sum_{v_iv_j\in E(G)} f\left(d(v_i),d(v_j)\right)\leq m\left[\frac{2(\Delta-1)}{\Delta^2}\right]^{\alpha} \tag{1}$$

with equality iff $d(v_i)=d(v_j)=\Delta$ for any $v_iv_j\in E(G)$.

If $\alpha>0$, then let $\Delta=d(v_k)$ for some $v_k\in V(G)$, where $1\leq k\leq n$,

$$\sum_{v_i:v_iv_k\in E(G)} f\left(d(v_i),d(v_k)\right)=\sum_{v_i:v_iv_k\in E(G)}\left[\frac{1}{\Delta}+\frac{1}{d(v_i)}\left(1-\frac{2}{\Delta}\right)\right]^{\alpha}\leq \Delta\left(\frac{\Delta+\delta-2}{\Delta\delta}\right)^{\alpha} \tag{2}$$

with equality iff $d(v_i)=\delta$ for every $v_i\in N(v_k)$. Recall that $\delta\geq 2$. By Lemma 2.2 (iii), if $\alpha>0$, then for any $v_iv_j\in E(G)$,

$$f\left(d(v_i),d(v_j)\right)\leq f(\delta,\delta)=\left[\frac{2(\delta-1)}{\delta^2}\right]^{\alpha}$$

with equality iff $d(v_i)=d(v_j)=\delta$. Thus

$$\sum_{\substack{v_i:v_iv_j\in E(G)\\ i,j\neq k}} f\left(d(v_i),d(v_j)\right)\leq (m-\Delta)\left[\frac{2(\delta-1)}{\delta^2}\right]^{\alpha} \tag{3}$$

with equality iff $d(v_i)=d(v_j)=\delta$ for any $v_iv_j\in E(G)$.

Note that G has clique number q. Then G is a K_{q+1}-free graph. By Lemma 1,

$$m\leq \frac{n^2(q-1)}{2q} \tag{4}$$

with equality iff G is a balanced complete q-partite graph.

If $\alpha<0$, then by inequalities (1) and (4),

$$ABC_{\alpha}(G)\leq \frac{n^2(q-1)}{2q}\left[\frac{2(\Delta-1)}{\Delta^2}\right]^{\alpha}$$

with equality iff G is a balanced complete q-partite graph.

If $\alpha>0$, then by inequalities (2) , (3) and (4),

$$\begin{aligned}ABC_{\alpha}(G)&\leq \sum_{v_iv_j\in E(G)} f\left(d(v_i),d(v_j)\right)\\ &\leq \sum_{v_i:v_iv_k\in E(G)} f\left(d(v_i),d(v_k)\right)+\sum_{\substack{v_i:v_iv_j\in E(G)\\ i,j\neq k}} f\left(d(v_i),d(v_j)\right)\\ &\leq \Delta\left(\frac{\Delta+\delta-2}{\Delta\delta}\right)^{\alpha}+(m-\Delta)\left[\frac{2(\delta-1)}{\delta^2}\right]^{\alpha}\\ &\leq \Delta\left(\frac{\Delta+\delta-2}{\Delta\delta}\right)^{\alpha}+\left(\frac{n^2(q-1)}{2q}-\Delta\right)\left[\frac{2(\delta-1)}{\delta^2}\right]^{\alpha}\end{aligned}$$

with equality iff G is a balanced complete q-partite graph. ■

By Lemma 2.2 (iii), the following conclusion directly comes from Theorem 3.

Corollary 4 Let G be a connected graph of order n with clique number q. If $\alpha < 0$, then

$$ABC_\alpha(G) \le \frac{n^2(q-1)}{2q}\left[\frac{2(\Delta-1)}{\Delta^2}\right]^\alpha;$$

If $\alpha > 0$, then for $\delta \ge 2$,

$$ABC_\alpha(G) \le \frac{n^2(q-1)}{2q}\left[\frac{2(\delta-1)}{\delta^2}\right]^\alpha$$

with equalities iff G is a balanced complete q-partite graph.

III. Results for trees with given pendant number

For a positive integer $n \ge 4$, let $\mathcal{T}_n$ be the set of trees of order n. Denote by S_n the star of order n. The number of pendant vertices in a graph is called its pendant number. For a positive integer $2 \le p \le n-2$, let $\mathcal{T}_{n,p}$ be the set of trees of order n with pendant number p. Let $S_{n,p}$ be the tree formed from the path on $n-p+1$ vertices by attaching $p-1$ pendant vertices to an end vertices. For a tree T and a vertex set $V_0 \subseteq V(T)$, $T - V_0$ denotes the tree formed from T by deleting the vertices V_0 and their incident edges. Let $N_1(v) = \{u|uv \in E(T), d(u) = 1\}$.

The following conclusion directly comes from Lemma 2.

Lemma 5 For a fixed integer $k \ge 1$, let $g(x,y) = \left(\frac{x+y-2}{xy}\right)^\alpha - \left[\frac{x+y-k-2}{x(y-k)}\right]^\alpha$, where $x \ge 0$ and $y \ge k$. Then

(i) If $\alpha < 0$, then $g(1,y) < 0$. If $\alpha > 0$, then $g(1,y) > 0$.
(ii) $g(2,y) = 0$.
(iii) If $\alpha < 0$, then $g(x,y) > 0$ for $x \ge 3$. If $\alpha > 0$, then $g(x,y) < 0$ for $x \ge 3$.

Lemma 6 Let $T \in \mathcal{T}_{n,p}$ and $v \in V(T)$, *where* $N_1(v) \ne \emptyset$. Then for $\alpha < 0$,

$$ABC_\alpha(T) - ABC_\alpha(T - N_1(v)) \ge |N_1(v)|\left(\frac{p-1}{p}\right)^\alpha$$

with equality iff $d(v) = p$ and $d(w) = 2$ for any $w \in N(v) \setminus N_1(v)$.

Proof. Let $v \in V(T)$, *where* $N_1(v) \ne \emptyset$. Clearly, $d(v) \ge 2$. Since $2 \le p \le n-2$, $N(v) \setminus N_1(v)$ contains one vertex of degree at least two. Let $|N(v) \setminus N_1(v)| = k$. By Lemma 5 (ii) and (iii),

$$\sum_{w \in N(v)\setminus N_1(v)} g(d(w), d(v)) \ge 0.$$

Thus

$$\begin{aligned} ABC_\alpha(T) - ABC_\alpha(T - N_1(v)) &= |N_1(v)| f\big(1, d(v)\big) + \sum_{w \in N(v)\setminus N_1(v)} g(d(w), d(v)) \\ &\ge |N_1(v)| f\big(1, d(v)\big) \\ &\ge |N_1(v)|\left(\frac{d(v)-1}{d(v)}\right)^\alpha \end{aligned}$$

with equalities iff $d(w) = 2$ for any $w \in N(v) \setminus N_1(v)$. Since T has p pendant vertices, $d(v) \le p$. Note that $\left(\frac{x-1}{x}\right)^\alpha$ is decreasing for $x \ge 2$. Thus

$$ABC_\alpha(T) - ABC_\alpha(T - N_1(v)) \ge |N_1(v)|\left(\frac{p-1}{p}\right)^\alpha$$

with equality iff $d(v) = p$ and $d(w) = 2$ for any $w \in N(v) \setminus N_1(v)$. ■

Theorem 7 Let $T \in \mathcal{T}_{n,p}$. Then for $\alpha < 0$,

$$ABC_\alpha(T) \ge (p-1)\left(\frac{p-1}{p}\right)^\alpha + (n-p)\left(\frac{1}{2}\right)^\alpha$$

with equality iff $T = S_{n,p}$.

Proof. If $p = 2$, then $T = P_n = S_{n,2}$. The result holds. Assume that $p \ge 3$.

Take a vertex $v \in V(T)$ such that $N_1(v) \ne \emptyset$ and $d(v) \ge 3$ (If possible). By Lemma 6,

$$ABC_\alpha(T) \ge ABC_\alpha(T - N_1(v)) + |N_1(v)|\left(\frac{p-1}{p}\right)^\alpha \quad (5)$$

with equality iff $d(v) = p$ and $d(u) = 2$ for any $u \in N(v) \setminus N_1(v)$ in T.

It is clear that $T - N_1(v)$ is a tree with p_1 pendant vertices, where $p - |N_1(v)| \le p_1 \le p - |N_1(v)| + 1$. Note that $|V(T - N_1(v))| = n - |N_1(v)|$ and $|E(T - N_1(v))| = n - |N_1(v)| - 1$. Let $T_1 = T - N_1(v)$. Similarly, Take a vertex $w \in V(T_1)$ such that $N_1(w) \ne \emptyset$ and $d(w) \ge 3$ (If possible). By Lemma 6,

$$ABC_\alpha(T_1) \ge ABC_\alpha(T_1 - N_1(w)) + |N_1(w)|\left(\frac{p_1-1}{p_1}\right)^\alpha \quad (6)$$

with equality iff $d(w) = p_1$ and $d(u) = 2$ for any $u \in N(w) \setminus N_1(w)$ in T_1. Since $p_1 \le p$, then $\left(\frac{p_1-1}{p_1}\right)^\alpha \ge \left(\frac{p-1}{p}\right)^\alpha$. Let $T_2 = T_1 - N_1(w)$. By inequalities (5) and (6),

$$ABC_\alpha(T) \geq ABC_\alpha(T_2) + (n - |V(T_2)|)\left(\frac{p-1}{p}\right)^\alpha \qquad (7)$$

Continue the above operation until the final graph T^* has no vertex u such that $N_1(u) \neq \emptyset$ and $d(u) \geq 3$. Also note that each edge of T^* has at least an end of degree two. Thus

$$\begin{aligned} ABC_\alpha(T) &\geq ABC_\alpha(T^*) + (n - |V(T^*)|)\left(\frac{p-1}{p}\right)^\alpha \\ &\geq (|V(T^*)| - 1)\left(\frac{1}{2}\right)^\alpha + (n - |V(T^*)|)\left(\frac{p-1}{p}\right)^\alpha \qquad (8) \end{aligned}$$

Since $2 \leq p \leq n-2$, then the number of the edges with weigh $\frac{p-1}{p}$ with respect to the general atom-bond connectivity index in T is less than or equal to $p-1$. Hence $-|V(T^*)| \leq p-1$, that is, $|V(T^*)| \geq n-p+1$. By lemma 2 (i), $\left(\frac{p-1}{p}\right)^\alpha - \left(\frac{1}{2}\right)^\alpha < 0$. Thus

$$ABC_\alpha(T) \geq (p-1)\left(\frac{p-1}{p}\right)^\alpha + (n-p)\left(\frac{1}{2}\right)^\alpha$$

with equality iff $T = S_{n,p}$. ■

Lemma 8 Let $T \in \mathcal{T}_{n,p}$, $uv \in E(T)$ and $d(u) = 1$. Then for $\alpha > 0$,

$$ABC_\alpha(T) - ABC_\alpha(T-u) \leq (p-1)\left(\frac{p-1}{p}\right)^\alpha - (p-2)\left(\frac{p-2}{p-1}\right)^\alpha$$

with equality iff $T = S_{n,p}$ and $d(v) = p$.

Proof. Take $uv \in E(T)$ and $d(u) = 1$. Clearly, $2 \leq d(v) \leq p$. Since $2 \leq p \leq n-2$, $N(v) \setminus \{u\}$ contains one vertex of degree at least two. By Lemma 5 (i), (ii) and (iii),

$$\begin{aligned} ABC_\alpha(T) - ABC_\alpha(T-u) &= f(1, d(v)) + \sum_{w \in N(v)\setminus\{u\}} g(d(w), d(v)) \\ &\leq f(1, d(v)) + g(2, d(w)) + (d(v) - 2)g(1, d(v)) \\ &= \left(\frac{d(v)-1}{d(v)}\right)^\alpha + (d(v) - 2)\left[\left(\frac{d(v)-1}{d(v)}\right)^\alpha - \left(\frac{d(v)-2}{d(v)-1}\right)^\alpha\right] \\ &= (d(v) - 1)\left(\frac{d(v)-1}{d(v)}\right)^\alpha - (d(v) - 2)\left(\frac{d(v)-2}{d(v)-1}\right)^\alpha \qquad (9) \end{aligned}$$

with equality iff $N(v)$ has exactly one vertex of degree two and $|N(v)| - 1$ vertices of degree one. Let $F(x) = (x-1)\left(\frac{x-1}{x}\right)^\alpha - (x-2)\left(\frac{x-2}{x-1}\right)^\alpha$ for $x \geq 2$, where $\alpha > 0$. Then

$$\begin{aligned} \frac{dF(x)}{dx} &= \left(\frac{x-1}{x}\right)^\alpha + \alpha(x-1)\left(\frac{x-1}{x}\right)^{\alpha-1}\frac{1}{x^2} - \left(\frac{x-2}{x-1}\right)^\alpha - \alpha(x-2)\left(\frac{x-2}{x-1}\right)^{\alpha-1}\frac{1}{(x-1)^2} \\ &= \left(\frac{x-1}{x}\right)^\alpha\left(1 + \frac{\alpha}{x}\right) - \left(\frac{x-2}{x-1}\right)^\alpha\left(1 + \frac{\alpha}{x-1}\right) \qquad (10) \end{aligned}$$

Let $H(x) = \left(\frac{y-1}{y}\right)^\alpha\left(1 + \frac{\alpha}{y}\right)$ for $y \geq 1$, where $\alpha > 0$. Then

$$\frac{dH(y)}{dy} = \alpha\left(\frac{y-1}{y}\right)^{\alpha-1}\frac{1}{y^2}\left(1 + \frac{\alpha}{y}\right) - \left(\frac{y-1}{y}\right)^\alpha\frac{\alpha}{y^2} = \left(\frac{y-1}{y}\right)^{\alpha-1}\frac{\alpha(1+\alpha)}{y^3}.$$

Clearly, $\frac{dH(y)}{dy} \geq 0$. Thus $H(y)$ is increasing for $y \geq 1$. By the equation (10), $\frac{dF(x)}{dx} \geq 0$ and hence $F(x)$ is increasing for $x \geq 2$. Recall that $2 \leq d(v) \leq p$. By the inequality (9),

$$ABC_\alpha(T) - ABC_\alpha(T-u) \leq (p-1)\left(\frac{p-1}{p}\right)^\alpha - (p-2)\left(\frac{p-2}{p-1}\right)^\alpha$$

with equality iff $T = S_{n,p}$ and $d(v) = p$. ■

Theorem 9 Let $T \in \mathcal{T}_{n,p}$. Then for $\alpha > 0$,

$$ABC_\alpha(T) \leq (p-1)\left(\frac{p-1}{p}\right)^\alpha + (n-p)\left(\frac{1}{2}\right)^\alpha$$

with equality iff $T = S_{n,p}$.

Proof. We argue by induction on n. It is trivial for $n = 4$. Suppose that $n \geq 5$ and it holds for trees with order $n-1$. Let $T \in \mathcal{T}_{n,p}$, $uv \in E(T)$ and $d(u) = 1$. Now we consider the following two cases.

Case 1 $d(u) = 2$.

Let $N(v) = \{u, w\}$. Then $d(w) \geq 2$ and

$$ABC_\alpha(T) - ABC_\alpha(T-u) = \left(\frac{1}{2}\right)^\alpha + \left(\frac{1}{2}\right)^\alpha - \left(\frac{d(w)-1}{d(w)}\right)^\alpha \leq \left(\frac{1}{2}\right)^\alpha$$

with equality iff $d(w) = 2$. Note that $T - u$ contains p pendant vertices. If $p = n-2$, then $T - u = S_{n-1}$ and hence $T = S_{n,n-2}$. If $p \leq n-3$, then by the induction hypothesis,

$$\begin{aligned} ABC_\alpha(T) &\le ABC_\alpha(T-u) + \left(\frac{1}{2}\right)^\alpha \\ &\le (p-1)\left(\tfrac{p-1}{p}\right)^\alpha + (n-1-p)\left(\tfrac{1}{2}\right)^\alpha + \left(\tfrac{1}{2}\right)^\alpha \\ &= (p-1)\left(\tfrac{p-1}{p}\right)^\alpha + (n-p)\left(\tfrac{1}{2}\right)^\alpha \end{aligned}$$

with equality iff $T-u=S_{n-1,p}$ and $d(w)=2$, i.e., $T=S_{n,p}$.

Case 2 $d(u)\ge 3$.

Note that $p\ge 3$ and $T-u$ contains $p-1$ pendant vertices. By Lemma 8,

$$ABC_\alpha(T) \le ABC_\alpha(T-u) + (p-1)\left(\tfrac{p-1}{p}\right)^\alpha - (p-2)\left(\tfrac{p-2}{p-1}\right)^\alpha.$$

By the induction hypothesis,

$$\begin{aligned} ABC_\alpha(T) &\le (p-2)\left(\tfrac{p-2}{p-1}\right)^\alpha + [n-1-(p-1)]\left(\tfrac{1}{2}\right)^\alpha + (p-1)\left(\tfrac{p-1}{p}\right)^\alpha - (p-2)\left(\tfrac{p-2}{p-1}\right)^\alpha \\ &= (p-1)\left(\tfrac{p-1}{p}\right)^\alpha + (n-p)\left(\tfrac{1}{2}\right)^\alpha \end{aligned}$$

with equality iff $T-u=S_{n-1,p-1}$ and the degree of v in $T-u$ is $p-1$, i.e., $T=S_{n,p}$. ■

Lemma 10 Let $h(p)=(p-1)\left(\frac{p-1}{p}\right)^\alpha+(n-p)\left(\frac{1}{2}\right)^\alpha$ for $2\le p\le n-2$. Then if $\alpha<0$, then $h(p)$ is decreasing; if $\alpha>0$, then $h(p)$ is increasing.

Proof. Consider the derivative of $h(p)$, we have

$$\frac{\mathrm{d}h(p)}{\mathrm{d}p} = \frac{\alpha}{p^2}\left(\frac{p-1}{p}\right)^{\alpha-1}(p-1) + \left(\frac{p-1}{p}\right)^\alpha - \left(\frac{1}{2}\right)^\alpha = \left(1+\frac{\alpha}{p}\right)\left(\frac{p-1}{p}\right)^\alpha - \left(\frac{1}{2}\right)^\alpha \tag{11}$$

If $\alpha<0$, then $1+\frac{\alpha}{p}<1$ and $\left(\frac{p-1}{p}\right)^\alpha\le\left(\frac{1}{2}\right)^\alpha$. By the equation (11), $\frac{\mathrm{d}h(p)}{\mathrm{d}p}<0$. Thus $h(p)$ is decreasing. If $\alpha>0$, then $1+\frac{\alpha}{p}>1$ and $\left(\frac{p-1}{p}\right)^\alpha\ge\left(\frac{1}{2}\right)^\alpha$. By the equation (11), $\frac{\mathrm{d}h(p)}{\mathrm{d}p}>0$. Thus $h(p)$ is increasing. ■

By Theorem 7, Theorem 9 and Lemma 10, we have the following.

Corollary 11 For $n\ge 6$, let $T\in\mathcal{T}_n$. Then

(i) If $\alpha<0$ and $T\in\mathcal{T}_n\setminus\{S_{n,n-2},S_n\}$, then

$$ABC_\alpha(T) > ABC_\alpha(S_{n,n-2}) > ABC_\alpha(S_n).$$

(ii) If $\alpha>0$ and $T\in\mathcal{T}_n\setminus\{S_{n,n-2},S_n\}$, then

$$ABC_\alpha(T) < ABC_\alpha(S_{n,n-2}) < ABC_\alpha(S_n).$$

Proof. Let T be a tree with p pendant vertices, where $2\le p\le n-2$. If $\alpha<0$, then by Theorem 7 and Lemma 10, $ABC_\alpha(T)\ge h(n-2)$ with equality iff $T=S_{n,n-2}$. Note that

$$ABC_\alpha(S_{n,n-2}) = h(n-2) = \left(\tfrac{n-3}{n-2}\right)^\alpha(n-3)+2\left(\tfrac{1}{2}\right)^\alpha > \left(\tfrac{n-2}{n-1}\right)^\alpha(n-1) = ABC_\alpha(S_n).$$

Thus (i) holds. If $\alpha>0$, then similar as the case of $\alpha<0$, by Theorem 9 and Lemma 10, (ii) also holds. ■

IV. Conclusion

This paper obtains on some bounds of the general atom-bond connectivity index for connected graphs with given clique number and trees with given pendant number, and characterize the corresponding extremal graphs. Moreover, among the trees with order $n\ge 6$, we determine such trees with the minimum and second minimum general atom-bond connectivity index(ABC_α) for $\alpha<0$, and the maximum and second maximum general atom-bond connectivity index(ABC_α) for $\alpha>0$. As a follow-up of this study, characterizing such graphs with the maximum and minimum general atom-bond connectivity index is an interesting work.

Acknowledgment

This work was supported by the department of education of Guangdong province natural science foundation of China(Grant No. 2020KTSCX078), the startup project of doctor scientific research of Hanshan Normal University(Grant No. QN202024) .

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