A New Type of Generalized Difference Sequence Spaces of Fuzzy Numbers Defined By Modulas Function

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ABSTRACT: In this article we introduce and study the sequence space \( w^p(\Delta^p, f, p) \), \( \Delta^p \) – summable sequence of fuzzy numbers, \( \Delta^p \) – statistical convergent and also \( \Delta^p \) – pre-Cauchy sequences of fuzzy numbers by using modulas function. Further we show that \( w^p(\Delta^p, f, p) \) is a complete metric space.

KEYWORDS: Sequence of fuzzy numbers; Difference sequence; modulas function , statistically convergent; pre-Cauchy sequences of fuzzy numbers , modulas function.

I. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [25] and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [7] where it was shown that every convergent sequence is bounded. Nanda [9] studied the spaces of bounded and convergent sequence of fuzzy numbers and showed that they are complete metric spaces. In [13] Savaş studied the space \( m(\Delta) \), which we call the space of \( \Delta \)-bounded sequence of fuzzy numbers and showed that this is a complete metric space.

Let \( D \) denote the set of all closed and bounded intervals \( X = [a_1, b_1] \) on the real line \( R \). For \( X = [a_1, b_1] \in D \) and \( Y = [a_2, b_2] \in D \), define \( d(X, Y) \) by

\[
d(X, Y) = \max (|a_1 - b_1|, |a_2 - b_2|).
\]

It is known that \( (D, d) \) is a complete metric space.

A fuzzy real number \( X \) is a fuzzy set on \( R \) i.e. a mapping \( X : R \rightarrow L([0,1]) \) associating each real number \( t \) with its grade of membership \( X(t) \).

The \( \alpha \)-level set \( \{X \}^\alpha \) set of a fuzzy real number \( X \) for \( 0 < \alpha \leq 1 \), defined as

\[
\{X \}^\alpha = \{ t \in R : X(t) \geq \alpha \}.
\]

A fuzzy real number \( X \) is called convex, if \( X(t) \geq X(s) \wedge X(r) = \min (X(s), X(r)) \), where \( s < t < r \).

If there exists \( t_0 \in R \) such that \( X(t_0) = 1 \), then the fuzzy real number \( X \) is called normal.

A fuzzy real number \( X \) is said to be upper semi-continuous if for each \( \varepsilon > 0 \), \( X^{-1} ([0, a + \varepsilon]) \), for all \( a \in L \) is open in the usual topology of \( R \).

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by \( L(R) \).

The absolute value \( |X| \) of \( X \in L(R) \) is defined as (see for instance Kaleva and Seikkala [2])

\[
|X|(t) = \begin{cases} 
\max \{ X(t), X(-t) \} , & \text{if } t > 0 \\
0, & \text{if } t < 0.
\end{cases}
\]

Let \( \bar{d} : L(R) \times L(R) \rightarrow R \) be defined by

\[
\bar{d} (X, Y) = \sup_{0 \leq \alpha \leq 1} d \left( \{X \}^\alpha \cap \{Y \}^\alpha \right).
\]

Then \( \bar{d} \) defines a metric on \( L(R) \).

For \( X, Y \in L(R) \) define

\[
X \leq Y \text{ iff } X^a \leq Y^a \text{ for any } a \in [0, 1].
\]

A subset \( E \) of \( L(R) \) is said to be bounded above if there exists a fuzzy number \( M \), called an upper bound of \( E \), such that \( X \leq M \) for every \( X \in E \). \( M \) is called the least upper bound or supremum of \( E \) if \( M \) is an upper
bound and $M$ is the smallest of all upper bounds. A lower bound and the greatest lower bound or infimum are defined similarly. $E$ is said to be bounded if it is both bounded above and bounded below.

II. DEFINITIONS AND BACKGROUND

A sequence $X = (X_k)$ of fuzzy numbers is a function $X$ from the set $N$ of all positive integers into $L(R)$. The fuzzy number $X_k$ denotes the value of the function at $k \in N$ and is called the $k$-th term or general term of the sequence.

Definition 2.1: A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy number $X_0$, written as $\lim_{k \to \infty} X_k = X_0$, if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that $d(X_k, X_0) < \varepsilon$ for $k > n_0$.

Definition 2.2: The set of convergent sequences is denoted by $cF$. A sequence $X = (X_k)$ of fuzzy numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that $d(X_k, X_l) < \varepsilon$ for $k, l > n_0$.

Definition 2.3: A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded and the set of bounded sequences is denoted by $F\infty$. The notion of difference sequence of complex terms was introduced by Kizmaz [6]. This notion was further generalized by Et and Colak [2], Tripathy and Esi [16], Tripathy, Esi and Tripathy [17] and many others.

The idea of the statistical convergence of sequences was introduced by Fast [3] and Schoenberg [12] independently in order to extend the notion of convergence of sequences. It is also found in Zygmund [26]. Later on it was linked with summability by Fridy and Orhan [4], Maddox [8], Rath and Tripathy [11] and many others. In [10] Nuray and Savaş extended the idea to sequences of fuzzy numbers and discussed the concept of statistically Cauchy sequences of fuzzy numbers. In this article we extend these notions to difference sequences of fuzzy numbers.

The natural density of a set $K$ of positive integers is denoted by $\delta(K)$ and defined by

$$\delta(K) = \lim_{n \to \infty} \frac{1}{n} \text{card}\{k \leq n : k \in K\}$$

Definition 2.4: If a sequence $X$ satisfies a property $P$ for almost all $k$ except a set of natural density zero, then we say that $X$ satisfies $P$ for almost all $k$ and we write $a.a.k.$.

Definition 2.5: A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number $X_0$ if for every $\varepsilon > 0$, $\lim_{n \to \infty} \frac{1}{n} \text{card}\{k \leq n : d(X_k, X_0) \geq \varepsilon\} = 0$. We write $st\text{-lim} X_k = X_0$.

Throughout the article we denote by $w'$ the set of all sequences of fuzzy numbers.

Definition 2.6: A sequence $(X_k)$ of fuzzy numbers is said to be double $\Delta^r_s$-convergent to a fuzzy number $X_0$ if for each $\varepsilon > 0$ there exist $k_0 \in N$ such that

$$d(\Delta^r_sX_k, X_0) < \varepsilon \quad \text{for all} \quad k \geq k_0.$$

We write $\lim_{k \to \infty} \Delta^r_sX_k = X_0$ where $r$ and $s$ are two non-negative integers and

$$\Delta^r_sX_k = \Delta^{r-1}_sX_k - \Delta^{r-1}_sX_{k+r}, \text{ and } \Delta^r_sX_k = X_k \text{ for all } k \in N,$$

which is equivalent to the following binomial representation:

$$\Delta^r_sX_k = \sum_{i=0}^{s} (-1)^i \binom{s}{i} X_{k+ir}.$$

We recall that a modulus function $f$ is a function from $[0, \infty)$ to $[0, \infty)$ such that:

(i) $f(x) = 0$ if and only if $x = 0$.
(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$.
(iii) $f$ is increasing.
(iv) $f$ is continuous from the right at 0.

It follows that $f$ must be continuous everywhere on $[0, \infty)$ and a modulus function may be bounded or not bounded. Ruckle [24], Maddox [18], Srivastava and Mohanta [15], used modulus function $f$ to construct some sequence spaces, subsequently many authors.
A metric $\bar{d}$ on $L(R)$ is said to be translation invariant if $\bar{d}(X + Y, Y + Z) = \bar{d}(X, Y)$ for all fuzzy numbers $X$, $Y$, $Z$.

Let $(E_k, \bar{d}_k)$ be a sequence of fuzzy linear metric spaces under the translation invariant metrics $\bar{d}_k$s such that $E_k \subseteq E_{k+1}$ for each $k \in N$ where $X_k = \left( (X_k)_{i=1}^{\infty} \right) \in E_k$. Define

$W(E) = \left\{ X = (X_k) : X_k \in E_k \text{ for each } k \in N \right\}$

$W(E)$ is a linear space of fuzzy numbers under coordinatewise addition and scalar multiplication. (see for instance [15])

Let $f$ be a modulus function and $p = (p_k)$ be a bounded sequence of positive real numbers. Also $r$ and $s$ be two non negative integers, we present the following new sequence space

$$w^f(\Delta^r, f, p) = \left\{ X = (X_k) \in W(E) : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left| f(\sup_k \bar{d}_k(\Delta^r_{X,k}, L_k)) \right|^{p_k} \right\}$$

where $X_{k,j} = \Delta^r_{X,k} - \Delta^r_{X,k+r}$ and $\Delta^r_{X,k} = X_{k,j}$ for all $j \in N$, which is equivalent to the following binomial representation

$$\Delta^r_{X,k,j} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} X_{k+r-i,j}.$$

**Definition 2.7:** A sequence $X = \left( (X_k)_{i=1}^{\infty} \right)$ of fuzzy numbers is said to be $\Delta^r f$-statistically convergent to a fuzzy number $L_k \in E_k$, $k \in N$ if for each $\varepsilon > 0$ such that

$$\lim_{n \to \infty} \text{card} \left\{ l \leq n : \sup_k \bar{d}_k(\Delta^r_{X,k,l}, L_k) \geq \varepsilon \right\} = 0$$

The set of all $\Delta^r f$-statistically convergent is denoted by $S^f(\Delta^r f)$.

**Definition 2.8:** A sequence $X = \left( (X_k)_{i=1}^{\infty} \right)$ of fuzzy numbers is said to be $\Delta^r f$-statistically Cauchy sequence, if for each $\varepsilon > 0$, there exists a positive integer $l_0$ such that

$$\lim_{n \to \infty} \text{card} \left\{ l \leq n : \sup_k \bar{d}_k(\Delta^r_{X,k,l}, \Delta^r_{X,k,l_0}) \geq \varepsilon \right\} = 0$$

**Definition 2.9:** A sequence $X = \left( (X_k)_{i=1}^{\infty} \right)$ of fuzzy numbers is said to be $\Delta^r f$-statistically pre-Cauchy sequence, if for all $\varepsilon > 0$,

$$\lim_{n \to \infty} \text{card} \{ (x, y) : x, y \leq n, \sup_k \bar{d}_k(\Delta^r_{X,k,x}, \Delta^r_{X,k,y}) \geq \varepsilon \} = 0$$

**Lemma 2.1:** If $\bar{d}$ is translation invariant then

(a) $\bar{d}(\Delta^r_{X,k,l} + \Delta^r_{Y,k,l}, 0) \leq \bar{d}(\Delta^r_{X,k,l}, 0) + \bar{d}(\Delta^r_{Y,k,l}, 0)$

(b) $\bar{d}(a\Delta^r_{X,k,l}, 0) \leq |a| \bar{d}(\Delta^r_{X,k,l}, 0), |a| > 1$.

**Lemma 2.2:** Let $(a_k)$ and $(b_k)$ be sequences of real or complex numbers and $(p_k)$ be a bounded sequence of positive real numbers, then

$$|a_k + b_k|^p \leq C(|a_k|^p + |b_k|^p)$$

and

$$|a|^p \leq \max \{1, |a|^\gamma - 1\}, C = \sup p_k, \gamma$$

where $C = \max \{1, |a|^{\gamma - 1}\}$.

**III. MAIN RESULTS**

**Theorem 3.1:** If $f$ be a modulus function and $0 < h = \inf p_k \leq p_k \leq \sup p_k = H$, then

$$w^f(\Delta^r, f, p) \subseteq S^f(\Delta^r f)$$

**Proof:** Let $\varepsilon > 0$ be given and $X = \left( (X_k)_{i=1}^{\infty} \right) \in w^f(\Delta^r, f, p)$. Then
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\[ \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) \right) \right)^{p_{k}} \]

\[ = \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) \right) \right)^{p_{k}} \]

\[ + \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) \right) \right)^{p_{k}} \]

\[ \geq \frac{M}{n} \text{card} \left\{ i \leq n : \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) \geq \varepsilon \right\} \]

where \( M = \min \left\{ f(\varepsilon)^{k}, f(\varepsilon)^{H} \right\} \). This follows that \( X \in S^{c}(\Delta^{\alpha}) \) and hence completes the proof.

**Theorem 3.2:** If \( f \) is bounded modulas function and \( X = \left( \left( X_{k,i} \right)_{i=1}^{\infty} \right) \) is statistically convergent, then \( X \in w^{c}(\Delta^{\alpha}, f, p) \).

**Proof:** Since \( f \) is bounded modulas function, therefore there exists an integer \( K \) such that \( f(\varepsilon) < K \). Let \( \varepsilon > 0 \) be given. Consider

\[ \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) \right) \right)^{p_{k}} \]

\[ = \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) \right) \right)^{p_{k}} \]

\[ + \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) \right) \right)^{p_{k}} \]

\[ \leq \max \left\{ \frac{1}{n} \text{card} \left\{ i \leq n : \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) \geq \varepsilon \right\} \middle| f(\varepsilon)^{k}, f(\varepsilon)^{H} \right\} \to 0 \text{ as } n \to \infty \]

Thus \( X \in w^{c}(\Delta^{\alpha}, f, p) \). This completes the proof.

**Theorem 3.3:** If a sequence \( X = \left( \left( X_{k,i} \right)_{i=1}^{\infty} \right) \) is \( \Delta^{\alpha} \) —stastically convergent, then it is \( \Delta^{\alpha} \) —statistically Cauchy sequence.

**Proof:** Since \( X \) is \( \Delta^{\alpha} \) —stastically convergent, so we have for each \( \varepsilon > 0 \),

\[ \lim_{n \to \infty} \frac{1}{n} \text{card} \left\{ i \leq n : \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) \geq \varepsilon \right\} = 0 \]

i.e

\[ \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i}, L_{k} \right) < \varepsilon \quad \text{a.a.l.} \]

We can choose \( l_{1} \in N \) such that

\[ \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i,l}, L_{k} \right) < \varepsilon \]

Now,

\[ \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i,l}, \Delta^{\alpha}_{k} X_{k,i,l} \right) \leq \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i,l}, L_{k} \right) + \sup_{k} \bar{d}_{k} \left( \Delta^{\alpha}_{k} X_{k,i,l}, L_{k} \right) \]

\[ < \varepsilon + \varepsilon = 2\varepsilon \quad \text{a.a.l.} \]

This implies that \( X \) is \( \Delta^{\alpha} \) —statistically Cauchy sequence.
Theorem 3.4: If a sequence \( X = \left( (X_{k,l})_{l=1}^{\infty} \right) \) is \( \Delta^p_k \)-statistically convergent, then it is \( \Delta^p_k \)-statistically bounded sequence.

Proof: Since \( X \) is \( \Delta^p_k \)-statistically convergent, so we have for each \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} \text{card} \{ 1 \leq n : \sup_{k} \bar{d}_k(\Delta^p_k X_{k,l}, L_k) \geq \epsilon \} = 0
\]

\( i.e \)

\[
\sup_{k} \bar{d}_k(\Delta^p_k X_{k,l}, L_k) < \epsilon.
\]

One can find \( \sup_{k} \bar{d}_k(L_k, \emptyset) < N \) (say). Then we have

\[
\sup_{k} \bar{d}_k(\Delta^p_k X_{k,l}, \emptyset) \leq \sup_{k} \bar{d}_k(\Delta^p_k X_{k,l}, L_k) + \sup_{k} \bar{d}_k(L_k, \emptyset) < \epsilon + N \quad a.a. k.
\]

Hence \( X \) is \( \Delta^p_k \)-statistically bounded sequence.

Remark: The converse of the above theorem is not true. To justify it, we consider the following example.

Example 3.1: Take \( f(k) = x, r = s = 1, p_k = 1 \) for each \( k \in N \) and \( E_k = L(k) \).

Define the sequence \( (X_k) \) as follows:

When \( k = 10^n \)

\[
X_k(t) = \begin{cases} 
1 + tk^2 & \text{if } -\frac{1}{k^2} \leq t \leq 0 \\
1 - tk^2 & \text{if } 0 \leq t \leq \frac{1}{k^2} \\
0 & \text{otherwise}
\end{cases}
\]

When \( k \neq 10^n \) and \( k \) is odd,

\[
X_k(t) = \begin{cases} 
(t + 5) & \text{if } -5 \leq t \leq -4 \\
-t - 3 & \text{if } -4 \leq t \leq -3 \\
0 & \text{otherwise}
\end{cases}
\]

When \( k \neq 10^n \) and \( k \) is even,

\[
X_k(t) = \begin{cases} 
(t - 3) & \text{if } 3 \leq t \leq 4 \\
-t + 5 & \text{if } 4 \leq t \leq 5 \\
0 & \text{otherwise}
\end{cases}
\]

Then,

\[
[X_k]^r = \begin{cases} 
\left[ \frac{a - 1}{k^2}, \frac{1 - a}{k^2} \right] & \text{when } k = 10^n \\
[-5 + a, -3 - a] & \text{when } k \neq 10^n \text{ and } k \text{ is odd} \\
[3 + a, 5 - a] & \text{when } k \neq 10^n \text{ and } k \text{ is even}
\end{cases}
\]

Therefore,

\[
[\Delta X_k]^r = \begin{cases} 
\left[ \frac{a - 1 + ak^2 + 3k^2}{k^2}, \frac{1 - a + 5k^2 - ak^2}{k^2} \right] & \text{when } k = 10^n \\
[-10 + 2a, -6 - 2a] & \text{when } k \neq 10^n \text{ and } k \text{ is odd} \\
[6 + 2a, 10 - 2a] & \text{when } k \neq 10^n \text{ and } k \text{ is even}
\end{cases}
\]

It follows that \( X \) is \( \Delta^p_k \)-statistically bounded but not \( X \) is \( \Delta^p_k \)-statistically convergent sequence.

Theorem 3.5: If \( X = \left( (X_{k,l})_{l=1}^{\infty} \right) \) is a sequence for which there exists a \( \Delta^p_k \)-statistically convergent sequence \( Y = \left( (Y_{k,l})_{l=1}^{\infty} \right) \) such that \( \Delta^p_k X_{k,l} = \Delta^p_k Y_{k,l} \) a.a. l. Then \( X \) is also \( \Delta^p_k \)-statistically convergent sequence.

Proof: Given that, \( \Delta^p_k X_{k,l} = \Delta^p_k Y_{k,l} \) a.a. l and \( Y \) is \( \Delta^p_k \)-statistically convergent sequence. Then
for each $\varepsilon > 0$ and each $n$, we have,
$$\{ l \leq n : \sup_k \bar{d}_k(\Delta^*_l X_{k,l}, L_k) \geq \varepsilon \} \cup \{ l \leq n : \Delta^*_l X_{k,l} \neq \Delta^*_l Y_{k,l} \}$$

$Y$ is $\Delta^*_l$—statistically convergent sequence, therefore the set $\{ l \leq n : \sup_k \bar{d}_k(\Delta^*_l X_{k,l}, L_k) \geq \varepsilon \}$ contains a fixed number $l_0 = l_0(\varepsilon)$. Then

$$\frac{1}{n} \text{card} \{ l \leq n : \sup_k \bar{d}_k(\Delta^*_l X_{k,l}, L_k) \geq \varepsilon \} \leq \frac{1}{n} + \frac{1}{n} \text{card} \{ l \leq n : \Delta^*_l X_{k,l} = \Delta^*_l Y_{k,l} \} \to 0 \text{ as } n \to \infty$$

This implies that $X$ is $\Delta^*_l$—statistically convergent.

**Theorem 3.6:** Let $X = (X_n)$ be a sequence of fuzzy number and $\Delta^*_l$—bounded. Then $X$ is $\Delta^*_l$—statistically pre-Cauchy sequence if and only if

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{x \leq y \leq n} f(\sup_k \bar{d}_k(\Delta^*_l X_{x,y}, \Delta^*_l X_{y,y})) = 0$$

where $f$ is bounded modulus function.

**Proof:** Let us first assume that,

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{x \leq y \leq n} f(\sup_k \bar{d}_k(\Delta^*_l X_{x,y}, \Delta^*_l X_{y,y})) = 0.$$

Given $\varepsilon > 0$ and for $n \in \mathbb{N}$, we have,

$$\frac{1}{n^2} \sum_{x \leq y \leq n} f(\sup_k \bar{d}_k(\Delta^*_l X_{x,y}, \Delta^*_l X_{y,y}))$$

$$= \frac{1}{n^2} \sum_{x \leq y \leq n} f(\sup_k \bar{d}_k(\Delta^*_l X_{x,y}, \Delta^*_l X_{y,y}))$$

$$+ \frac{1}{n^2} \sum_{x \leq y \leq n} f(\sup_k \bar{d}_k(\Delta^*_l X_{x,y}, \Delta^*_l X_{y,y}))$$

$$\geq \frac{1}{n^2} \text{card} \{ (x, y) : x, y \leq n, \sup_k \bar{d}_k(\Delta^*_l X_{x,y}, \Delta^*_l X_{y,y}) \geq \varepsilon \}$$

and hence $X$ is $\Delta^*_l$—statistically pre-Cauchy.

Conversely let $X$ is $\Delta^*_l$—statistically pre-Cauchy and $\varepsilon > 0$ be given. Choose $\lambda > 0$ such that $f(\lambda) < \frac{\varepsilon}{2}$. Since $f$ is bounded modulus function, therefore there exists an integer $M$ such that

$$f(\sup_k \bar{d}_k(\Delta^*_l X_{x,y}, \Delta^*_l X_{y,y})) < M.$$
By our assumption, i.e. there exists a positive integer $n_0$ such that,

$$\lim_{n \to \infty} \frac{1}{n^2} \text{card}\{(x, y) : x, y \leq n, \sup_k d_k(\Delta^f_{x}X_{k,x}, \Delta^f_{y}X_{k,y}) \geq \lambda\} = 0$$

i.e there exists a positive integer $n_0$ such that,

$$\frac{1}{n^2} \text{card}\{(x, y) : x, y \leq n, \sup_k d_k(\Delta^f_{x}X_{k,x}, \Delta^f_{y}X_{k,y}) \geq \lambda\} \leq \frac{\epsilon}{2M} \quad \forall \ n \geq n_0$$

i.e.

$$\frac{1}{n^2} \sum_{x,y \leq n} f(\sup_k d_k(\Delta^f_{x}X_{k,x}, \Delta^f_{y}X_{k,y})) \leq \epsilon \quad \forall \ n \geq n_0$$

Thus we have

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{x,y \leq n} f(\sup_k d_k(\Delta^f_{x}X_{k,x}, \Delta^f_{y}X_{k,y})) = 0$$

This completes the proof.

**Remark :** A sequence $X$ is $\Delta^f$ - statistically pre Cauchy but not $\Delta^f$ - statistically convergent. To justify it, we consider the following example.

**Example 3.2 :** Take $f(x) = x, r = 1, p_k = 1$ for each $k \in \mathbb{N}$ and $E_k = L(\mathbb{R})$. Consider the sequence $X = (X_k)$ given as follows:

When $k$ is even,

$$X_k(t) = \begin{cases} t - 3 & \text{if } 3 \leq t \leq 4 \\ -t + 5 & \text{if } 4 \leq t \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

When $k$ is odd,

$$X_k(t) = \begin{cases} t + 5 & \text{if } -5 \leq t \leq -4 \\ -t - 3 & \text{if } -4 \leq t \leq -3 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$[\Delta^f_{x}X_k]^e = \begin{cases} [2^e(3 + a), 2^e(5 - a)] & \text{if } k \text{ is even} \\ [2^e(-5 + a), 2^e(-3 - a)] & \text{if } k \text{ is odd} \end{cases}$$

This implies that $X$ is $\Delta^f$ - statistically pre Cauchy but not $\Delta^f$ - statistically convergent.

**Theorem 3.7:** If $(p_k)$ be a bounded sequence of positive real numbers. Then the space $w^f(\Delta^f, f, p)$ is a linear space over the real field $\mathbb{R}$.

**Proof:** The proof is easy, so omitted.

**Theorem 3.8:** Let $(E_k, d_k)$ be a sequence of complete metric spaces and $(p_k)$ be a bounded sequence of positive real numbers such that $\inf p_k > 0$. Then the space $w^f(\Delta^f, f, p)$ is complete metric space under the metric $d$ defined by-
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\[ g(X, Y) = \sup_n \left( \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_k \delta_k (\Delta_{i} X_{k, l}, \Delta_{i} Y_{k, l}) \right) \right)^p \right) \]

**Proof:** It is easy to see that \( g \) is a metric on \( w^f (\Delta_{i} X, f, p) \). We just prove completeness. Let \( (X^{(i)}) \) be a Cauchy sequence in \( w^f (\Delta_{i} X, f, p) \), where \( X^{(i)} = \left( \left( X^{(i)}_{k, l} \right)_{k=l}^{\infty} \right)_{k=l}^{\infty} \in w^f (\Delta_{i} X, f, p) \) \( \forall \ i \in N \). Then we have,

\[ g(X^{(i)}, X^{(j)}) \to 0 \text{ as } i, j \to \infty \]

This implies

\[ \sup_n \left( \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_k \delta_k (\Delta_{i}^l X_{k, l}^{(i)}, \Delta_{i}^l X_{k, l}^{(j)}) \right) \right)^p \right) \to 0 \text{ as } i, j \to \infty \]

Since \( f \) is modulus function we have

\[ \sup_k \delta_k (\Delta_{i}^l X_{k, l}^{(i)}, \Delta_{i}^l X_{k, l}^{(j)}) \to 0 \text{ as } i, j \to \infty \text{ and for each } l = 1, 2, 3, ..., n \]

This follows that,

\[ \delta_k (\Delta_{i}^l X_{k, l}^{(i)}, \Delta_{i}^l X_{k, l}^{(j)}) \to 0 \text{ as } i, j \to \infty \text{ and for each } l = 1, 2, 3, ..., n \]

i.e. \( (\Delta_{i}^l X_{k, l}^{(i)}) \) is a Cauchy sequence in \( E_k \). Since \( E_k \) is complete so \( \Delta_{i}^l X_{k, l}^{(i)} \) is convergent in \( E_k \). For simplicity let

\[ \lim_{i=\infty} \Delta_{i}^l X_{k, l}^{(i)} = \sum_{u=0}^{s} (-1)^u \binom{s}{u} X_{k+s+u} = N_{k,l}, \text{say for each } k \geq 1. \]

Considering \( k = 1, 2, 3, ..., rs \) and \( l = 1, 2, 3, ..., s \), we can easily conclude that

\[ \lim_{i=\infty} X_{k, l}^{(i)} = X_{k,l} \text{ for } l = 1, 2, 3, ..., s. \]

Taking limit as \( f \to \infty \) in (3.1), we have

\[ \lim_{i=\infty} g(X^{(i)}, X) = 0 \]

Now it remains to show \( X \in w^f (\Delta_{i} X, f, p) \). From (3.2) we get,

\[ \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_k \delta_k (\Delta_{i}^l X_{k, l}^{(i)}, \Delta_{i}^l X_{k, l}^{(j)}) \right) \right)^p \to 0 \text{ as } i \to \infty \ \forall \ n \in N \]

Therefore for any \( \varepsilon > 0 \) there exists a positive integer \( i_0 \) such that,

\[ \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_k \delta_k (\Delta_{i}^l X_{k, l}^{(i)}, \Delta_{i}^l X_{k, l}^{(j)}) \right) \right)^p < \frac{\varepsilon}{3} \ \forall \ i \geq i_0 \text{ and } n \in N. \]

Now one can find for each \( n_2, n_1 \in N \) such that,

\[ \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_k \delta_k (\Delta_{i}^l X_{k, l}^{(i)}, L_k^{(i)}) \right) \right)^p < \frac{\varepsilon}{3} \ \forall \ n \geq n_2 \text{ and } L_k^{(i)} \in E_k. \]

and

\[ \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_k \delta_k (\Delta_{i}^l X_{k, l}^{(j)}, L_k^{(j)}) \right) \right)^p < \frac{\varepsilon}{3} \ \forall \ n \geq n_1 \text{ and } L_k^{(j)} \in E_k. \]

Take \( i, j \geq i_0 \) and \( n_2 = \max(n_2, n_1) \). Then,

\[ \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_k \delta_k (L_k^{(i)}, L_k^{(j)}) \right) \right)^p \leq C \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_k \delta_k (\Delta_{i}^l X_{k, l}^{(i)}, L_k^{(i)}) \right) \right)^p + C \frac{1}{n} \sum_{i=1}^{n} \left( f \left( \sup_k \delta_k (\Delta_{i}^l X_{k, l}^{(j)}, L_k^{(j)}) \right) \right)^p < Ce \ \forall \ i, j \geq n_2. \]

Since \( f \) is monotone function, we have

\[ \delta_k (L_k^{(i)}, L_k^{(j)}) < \varepsilon_1 \ \forall \ i, j \geq n_2 \]
i.e. \((f^{(i)}_{k})\) is Cauchy sequence in \(E_{k}\) which is complete. So let \(L^{(i)}_{k} \rightarrow L_{k}\) as \(i \rightarrow \infty\), therefore,
\[
\frac{1}{n} \sum_{i=1}^{n} \left( f(\sup_{i} \bar{a}_{k}(L^{(i)}_{k}, L_{k})) \right)^{p}\leq C \forall i \geq n_{2}
\]
Thus we get,
\[
\frac{1}{n} \sum_{i=1}^{n} \left( f(\sup_{i} \bar{a}_{k}(\Delta^{i}_{k}X_{k}, L_{k})) \right)^{p} \leq C \frac{1}{n} \sum_{i=1}^{n} \left( f(\sup_{i} \bar{a}_{k}(\Delta^{i}_{k}X_{k}, L_{k})) \right)^{p} + C \frac{1}{n} \sum_{i=1}^{n} \left( f(\sup_{i} \bar{a}_{k}(L^{(i)}_{k}, L_{k})) \right)^{p} \leq 2Ce \forall \geq n_{2}\]
Which implies that \(X \in w^{f}(\Delta^{i}_{k}, f, p)\). This completes the proof.

**Theorem 3.9:** Let \((p_{k})\) and \((q_{k})\) be two sequences of positive real numbers such that \(0 < p_{k} \leq q_{k}\) and the sequence \(\frac{q_{k}}{p_{k}}\) is bounded. Then \(w^{f}(\Delta^{i}_{k}, f, q) \subseteq w^{f}(\Delta^{i}_{k}, f, p)\).

**Proof:** Let \(X \in w^{f}(\Delta^{i}_{k}, f, q)\), therefore, 
\[
\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} \left( f(\sup_{i} \bar{a}_{k}(\Delta^{i}_{k}X_{k}, L_{k})) \right)^{q_{k}} = 0
\]
Take \(\alpha_{k} = f(\sup_{i} \bar{a}_{k}(\Delta^{i}_{k}X_{k}, L_{k}))^{q_{k}}\) and \(\gamma_{k} = \frac{q_{k}}{p_{k}}\) s.t \(0 < \gamma_{k} \leq 1\), define \(a_{k} = \alpha_{k} + b_{k}\) and \(\alpha_{k}^{\gamma_{k}} = a_{k}^{\gamma_{k}} + b_{k}^{\gamma_{k}}\), this implies that \(a_{k}^{\gamma_{k}} \leq a_{k} \leq a_{k}^{\gamma_{k}}\) and \(b_{k}^{\gamma_{k}} \leq b_{k}^{\gamma_{k}}\).
Then, 
\[
\frac{1}{n} \sum_{i=1}^{n} \left( f(\sup_{i} \bar{a}_{k}(\Delta^{i}_{k}X_{k}, L_{k})) \right)^{p} \leq \frac{1}{n} \sum_{i=1}^{n} \left( f(\sup_{i} \bar{a}_{k}(\Delta^{i}_{k}X_{k}, L_{k})) \right)^{p_{k}} + \frac{1}{n} \sum_{i=1}^{n} b_{k}^{\gamma_{k}} - 0 \text{ as } n \rightarrow \infty
\]
i.e.
\[
\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} \left( f(\sup_{i} \bar{a}_{k}(\Delta^{i}_{k}X_{k}, L_{k})) \right)^{p_{k}} = 0
\]
It follows that \(X \in w^{f}(\Delta^{i}_{k}, f, p)\). This completes the proof.

**REFERENCES**

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