On Nano Forms Of Weakly Open Sets

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ABSTRACT : The purpose of this paper is to define and study certain weak forms of nano-open sets namely, nano α-open sets, nano semi-open sets and nano pre-open sets. Various forms of nano α-open sets and nano semi-open sets corresponding to different cases of approximations are also derived.

KEYWORDS: Nanotopology, nano-open sets, nano interior, nano closure, nano α-open sets, nano semi-open sets, nano pre-open sets, nano regular open sets.

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I. INTRODUCTION

Njastad [5], Levine [2] and Mashhour et al [3] respectively introduced the notions of α-open, semi-open and pre-open sets. Since then these concepts have been widely investigated. It was made clear that each α-open set is semi-open and pre-open but the converse of each is not true. Njastad has shown that the family τ" of α-open sets is a topology on X satisfying τ ⊆ τ" . The families SO(X, τ ) of all semi-open sets and PO(X, τ ) of all pre-open sets in (X, τ ) are not topologies. It was proved that both SO(X, τ ) and PO(X, τ ) are closed under arbitrary unions but not under finite intersection. Lellis Thivagar et al [1] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower and upper approximations of X. The elements of a nano topological space are called the nano-open sets. He has also studied nano closure and nano interior of a set. In this paper certain weak forms of nano-open sets such as nano α-open sets, nano semi-open sets and nano pre-open sets are established. Various forms of nano α-open sets and nano semi-open sets under various cases of approximations are also derived. A brief study of nano regular open sets is also made.

II. PRELIMINARIES

Definition 2.1 A subset A of a space (X, τ ) is called
(ii) pre open [3] if A ⊆ Int (Cl (A)) .
(iii) α-open [4] if A ⊆ Int (Cl (Int (A))) .
(iv) regular open [4] if A = Int (Cl (A)) .

Definition 2.2 [6] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let X ⊆ U .

(i) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and its is denoted by Lₙ(x) . That is, Lₙ(x) = \{R(x) : R(x) ⊆ X \}, where R(x) denotes the equivalence class determined by x.

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by Uₙ(x) . That is, Uₙ(x) = \{R(x) : R(x) ∩ X = ∅ \}

(iii) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor...
as not-X with respect to R and it is denoted by $B_n(X)$. That is, $B_n(X) = U_n(X) - L_n(X)$.

**Property 2.3** [6] If $(U, R)$ is an approximation space and $X, Y \subseteq U$, then

(i) $L_n(X) \subseteq X \subseteq U_n(X)$.

(ii) $L_n(\emptyset) = U_n(\emptyset) = \emptyset$ and $L_n(U) = U_n(U) = U$.

(iii) $U_n(X \cup Y) = U_n(X) \cup U_n(Y)$.

(iv) $U_n(X \cap Y) \subseteq U_n(X) \cap U_n(Y)$.

(v) $L_n(X \cup Y) \supseteq L_n(X) \cup L_n(Y)$.

(vi) $L_n(X \cap Y) = L_n(X) \cap L_n(Y)$.

(vii) $L_n(X) \subseteq L_n(Y)$ and $U_n(X) \subseteq U_n(Y)$ whenever $X \subseteq Y$.

(viii) $U_n(X^\prime) = [L_n(X)]^\prime$ and $L_n(X^\prime) = [U_n(X)]^\prime$.

(ix) $U_n U_n(X) = L_n U_n(X) = U_n(X)$.

(x) $L_n L_n(X) = U_n L_n(X) = L_n(X)$.

**Definition 2.4** [1]: Let $U$ be the universe, $R$ be an equivalence relation on $U$ and $\tau_n(X) = \{U, \emptyset, L_n(X), U_n(X), B_n(X)\}$ where $X \subseteq U$. Then by propety 2.3, $\tau_n(X)$ satisfies the following axioms:

(i) $\forall U$ and $\emptyset \in \tau_n(X)$.

(ii) The union of the elements of any subcollection of $\tau_n(X)$ is in $\tau_n(X)$.

(iii) The intersection of the elements of any finite subcollection of $\tau_n(X)$ is in $\tau_n(X)$.

That is, $\tau_n(X)$ is a topology on $U$ called the nanotopology on $U$ with respect to $X$. We call $(U, \tau_n(X))$ as the nanotopological space. The elements of $\tau_n(X)$ are called as nano-open sets.

**Remark 2.5** [1] If $\tau_n(X)$ is the nano topology on $U$ with respect to $X$, then the set $B = \{U, L_n(X), U_n(X), B_n(X)\}$ is the basis for $\tau_n(X)$.

**Definition 2.6** [1] If $(U, \tau_n(X))$ is a nano topological space with respect to $X$ where $X \subseteq U$ and if $A \subseteq U$, then the nano interior of $A$ is defined as the union of all nano-open subsets of $A$ and it is denoted by $N \text{Int}(A)$. That is, $N \text{Int}(A)$ is the largest nano-open subset of $A$. The nano closure of $A$ is defined as the intersection of all nano closed sets containing $A$ and it is denoted by $N \text{Cl}(A)$. That is, $N \text{Cl}(A)$ is the smallest nano closed set containing $A$.

**Definition 2.7** [1] A nano topological space $(U, \tau_n(X))$ is said to be extremally disconnected, if the nano closure of each nano-open set is nano-open.

**III. NANO $\alpha$ - OPEN SETS**

Throughout this paper $(U, \tau_n(X))$ is a nano topological space with respect to $X$ where $X \subseteq U$, $R$ is an equivalence relation on $U$, $\mathcal{U}/R$ denotes the family of equivalence classes of $U$ by $R$.

**Definition 3.1** Let $(U, \tau_n(X))$ be a nano topological space and $A \subseteq U$. Then $A$ is said to be

(i) nano semi-open if $A \subseteq N \text{Cl}(N \text{Int}(A))$

(ii) nano pre-open if $A \subseteq N \text{Int}(N \text{Cl}(A))$

(iii) nano $\alpha$ - open if $A \subseteq N \text{Int}(N \text{Cl}(N \text{Hnt}(A))$.

$N \text{SO}(U, X), N \text{PO}(U, X)$ and $\tau_n^\alpha(X)$ respectively denote the families of all nano semi-open, nano pre-open...
and nano $\alpha$-open subsets of $U$.

**Definition 3.2** Let $(U, \tau_u(X))$ be a nanotopological space and $A \subseteq U$. $A$ is said to be nano $\alpha$-closed (respectively, nano semi-closed, nano pre-closed), if its complement is nano $\alpha$-open (nano semi-open, nano pre-open).

**Example 3.3** Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$. Then the nano topology, $	au_u(X) = \{U, \emptyset, \{a\}, \{b, d\}, \{a, b\}\}$. The nano closed sets are $U, \emptyset, \{a\}, \{b, d\}, \{a, b\}$ and $\{a, c\}$. Then, $NSO(U, X) = \{U, \emptyset, \{a\}, \{a, c\}, \{a, b\}, \{a, b, d\}\}$, $NPO(U, X) = \{U, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. We note that, $NSO(U, X)$ does not form a topology on $U$, since $\{a, c\}$ and $\{b, c, d\} \in NPO(U, X)$ but $\{a, c\} \cap \{b, c, d\} = \{c\} \notin NSO(U, X)$. Similarly, $NPO(U, X)$ is not a topology on $U$, since $\{a, b, c\} \cap \{a, c, d\} = \{a, c\} \notin NPO(U, X)$, even though $\{a, b, c\}$ and $\{a, c, d\} \in NPO(U, X)$. But the sets of $\tau_u(X)$ form a topology on $U$. Also, we note that $\{a, c\} \in NSO(U, X)$ but is not in $NPO(U, X)$ and $\{a, b\} \in NPO(U, X)$, but does not belong to $NSO(U, X)$. That is, $NSO(U, X)$ and $NPO(U, X)$ are independent.

**Theorem 3.4** If $A$ is nano-open in $(U, \tau_u(X))$, then it is nano $\alpha$-open in $U$.

**Proof:** Since $A$ is nano-open in $U$, $IntA = A$. Then $Cl(NIntA) = Cl(A) \supseteq A$. That is $A \subseteq Cl(NIntA)$. Therefore, $Int(NCl(NInt(A))) \subseteq NInt(NCl(Cl(A)))$. That is, $A \subseteq NInt(NCl(NInt(A)))$. Thus, $A$ is nano $\alpha$-open.

**Theorem 3.5** $\tau_u^a(X) \subseteq NSO(U, X)$ in a nanotopological space $(U, \tau_u(X))$.

**Proof:** If $A \in \tau_u^a(X)$, $A \subseteq NInt(NCl(NInt(A))) \subseteq NInt(NCl(Cl(A)))$ and hence $A \in NSO(U, X)$.

**Remark 3.6** The converse of the above theorem is not true. In example 3.3, $\{a, c\}$ and $\{b, c, d\}$ are nano-open but are not nano $\alpha$-open in $U$.

**Theorem 3.7** $\tau_u^a(X) \subseteq NPO(U, X)$ in a nanotopological space $(U, \tau_u(X))$.

**Proof:** If $A \in \tau_u^a(X)$, $A \subseteq NInt(NCl(NInt(A)))$. Since $NInt(A) \subseteq A$, $NInt(NCl(NInt(A))) \subseteq NInt(NCl(Cl(A)))$. That is, $A \subseteq NInt(NCl(Cl(A)))$. Therefore, $A \in NPO(U, X)$. That is, $\tau_u^a(X) \subseteq NPO(U, X)$.

**Remark 3.8** The converse of the above theorem is not true. In example 3.3, the set $\{b\}$ is nano pre-open but is not nano $\alpha$-open in $U$.

**Theorem 3.9** $\tau_u^a(X) = NSO(U, X) \cap NPO(U, X)$.

**Proof:** If $A \in \tau_u^a(X)$, then $A \in NSO(U, X)$ and $A \in NPO(U, X)$ by theorems 3.5 and 3.7 and hence $A \in NSO(U, X) \cap NPO(U, X)$. That is $\tau_u^a(X) \subseteq NSO(U, X) \cap NPO(U, X)$. Conversely, if $A \in NSO(U, X) \cap NPO(U, X)$, then $A \subseteq NInt(NCl(Cl(A)))$ and $A \subseteq NInt(NCl(NInt(A)))$. Therefore, $NInt(NCl(Cl(A))) \subseteq NInt(NCl(NCl(NInt(A)))) = NInt(NCl(NInt(Cl(A))))$. That is, $A \subseteq NInt(NCl(Cl(A)))$. Also $A \subseteq NInt(NCl(Cl(A))) \subseteq NInt(NCl(NInt(Cl(A))))$. Therefore, $A \subseteq NInt(NCl(NInt(A)))$. That is, $A \in \tau_u^a(X)$. Thus, $NSO(U, X) \cap NPO(U, X) \subseteq \tau_u^a(X)$. Therefore, $\tau_u^a(X) = NSO(U, X) \cap NPO(U, X)$.
Theorem 3.10: If, in a nano topological space \((U, \tau_u(X))\), \(L_u(X) = U_u(X) = X\), then \(U, \phi, L_u(X)\) and any set \(A \supset L_u(X)\) are the only nano-\(\alpha\)-open sets in \(U\).

Proof: Since \(L_u(X) = U_u(X) = X\), the nano topology, \(\tau_u(X) = \{U, \phi, L_u(X)\}\). Since any nano-open set is nano-\(\alpha\)-open, \(U, \phi\) and \(L_u(X)\) are nano \(\alpha\)-open in \(U\). If \(A \subseteq L_u(X)\), then \(\text{Int}\; (A) = \phi\), since \(\phi\) is the only nano-open subset of \(A\). Therefore \(\text{NCl}\; (\text{Int}\; (A)) = \phi\) and hence \(A\) is not nano \(\alpha\)-open. If \(A \supset L_u(X)\), \(L_u(X)\) is the largest nano-open subset of \(A\) and hence, \(\text{Int}\; (\text{NCl}\; (\text{NInt}\; (A))) = \text{NInt}\; (\text{NCl}\; (\text{NInt}\; (A)))\). Therefore, \(A\) is nano \(\alpha\)-open. Thus \(U, \phi, L_u(X)\) and any set \(A \supset L_u(X)\) are the only nano \(\alpha\)-open sets in \(U\), if \(L_u(X) = U_u(X)\).

Theorem 3.11: \(U, \phi, U_u(X)\) and any set \(A \supset U_u(X)\) are the only nano \(\alpha\)-open sets in a nano-topological space \((U, \tau_u(X))\) if \(L_u(X) = \phi\).

Proof: Since \(L_u(X) = \phi\), \(B_u(X) = U_u(X)\). Therefore, \(\tau_u(X) = \{U, \phi, U_u(X)\}\) and the members of \(\tau_u(X)\) are nano \(\alpha\)-open in \(U\). Let \(A \subset U_u(X)\). Then \(\text{NInt}\; (A) = \phi\) and hence \(\text{Int}\; (\text{NCl}\; (\text{NInt}\; (A))) = \phi\). Therefore \(A\) is not nano \(\alpha\)-open in \(U\). If \(A \supset U_u(X)\), then \(U_u(X)\) is the largest nano-open subset of \(A\) (unless, \(U_u(X) = U\), in case of which \(U\) and \(\phi\) are the only nano-open sets in \(U\)). Therefore, \(\text{NInt}\; (\text{NCl}\; (\text{NInt}\; (A))) = \text{NInt}\; (\text{NCl}\; (U_u(X))) = \text{NInt}\; (U)\) and hence \(A \subseteq \text{NInt}\; (\text{NCl}\; (\text{NInt}\; (A)))\). Thus, any set \(A \supset U_u(X)\) is nano \(\alpha\)-open in \(U\). Hence, \(U, \phi, U_u(X)\) and any superset of \(U_u(X)\) are the only nano \(\alpha\)-open sets in \(U\).

Theorem 3.12: If \(U_u(X) = U\) and \(L_u(X) \neq \phi\), in a nano topological space \((U, \tau_u(X))\), then \(U, \phi, L_u(X)\) and \(B_u(X)\) are the only nano \(\alpha\)-open sets in \(U\).

Proof: Since \(U_u(X) = U\) and \(L_u(X) \neq \phi\), the nano-open sets in \(U\) are \(U, \phi, L_u(X)\) and \(B_u(X)\) and hence they are nano \(\alpha\)-open also. If \(A = \phi\), then \(A\) is nano \(\alpha\)-open. Therefore, let \(A \neq \phi\). When \(A \subset L_u(X)\), \(\text{NInt}\; (A) = \phi\), since the largest open subset of \(A\) is \(\phi\) and hence \(A \subset \text{NInt}\; (\text{NCl}\; (\text{NInt}\; (A)))\), unless \(A\) is \(\phi\). That is, \(A\) is not nano \(\alpha\)-open in \(U\). When \(L_u(X) \subset A\), \(\text{NInt}\; (A) = L_u(X)\) and therefore, \(\text{Int}\; (\text{NCl}\; (\text{NInt}\; (A))) = \text{Int}\; (\text{NCl}\; (L_u(X))) = \text{NInt}\; (B_u(X))\). Similarly, it can be shown that any set \(A \subset B_u(X)\) and \(A \supset B_u(X)\) are not nano \(\alpha\)-open in \(U\). If \(A\) has at least one element each of \(L_u(X)\) and \(B_u(X)\), then \(\text{NInt}\; (A) = \phi\) and hence \(A\) is not nano \(\alpha\)-open in \(U\). Hence, \(U, \phi, L_u(X)\) and \(B_u(X)\) are the only nano \(\alpha\)-open sets in \(U\) when \(U_u(X) = U\) and \(L_u(X) \neq \phi\).

Corollary 3.13: \(\tau_u(X) = \tau_u^n(X)\), if \(U_u(X) = U\).

Theorem 3.14: Let \(L_u(X) \neq U_u(X)\) where \(L_u(X) \neq \phi\) and \(U_u(X) \neq U\) in a nano topological space \((U, \tau_u(X))\). Then \(U, \phi, L_u(X), B_u(X), U_u(X)\) and any set \(A \supset U_u(X)\) are the only nano \(\alpha\)-open sets in \(U\).

Proof: The nano topology on \(U\) is given by \(\tau_u(X) = \{U, \phi, L_u(X), B_u(X), U_u(X)\}\) and hence \(U, \phi, L_u(X), B_u(X)\) and \(U_u(X)\) are nano \(\alpha\)-open in \(U\). Let \(A \subseteq U\) such that \(A \supset U_u(X)\). Then
N Int (A) = Uₙ(X) and therefore, N Int (N Cl (Uₙ(X))) = N Int (U) = U. Hence, A ⊆ N Int (N Cl (N Int (A))). Therefore, any A ⊆ Uₙ(X) is nano α-open in U. When A ⊆ Lₙ(X), N Int (A) = φ and hence N Int (N Cl (N Int (A))) = φ. Therefore, A is not nano α-open in U. When A ⊆ Bₙ(X), N Int (A) = φ and hence A is not nano α-open in U. When A ⊆ Uₙ(X) such that A is neither a subset of Lₙ(X) nor a subset of Bₙ(X), then N Int (A) = φ and hence A is not nano α-open in U. Thus, A \not\subseteq Uₙ(X) and any set A \not\subseteq Uₙ(X) are the only nano α-open sets in U.

IV. FORMS OF NANO SEMI-OPEN SETS AND NANO REGULAR OPEN SETS

In this section, we derive forms of nano semi-open sets and nano regular open sets depending on various combinations of approximations.

Remark 4.1 U, φ are obviously nano semi-open, since N Cl (N Int (U)) = U and N Cl (N Int (φ)) = φ.

Theorem 4.2 If, in a nano topological space (X, τₙ(X)), Uₙ(X) = Lₙ(X), then φ and sets A such that A \supseteq Lₙ(X) are the only nano semi-open subsets of U.

Proof: τₙ(X) = {U, φ, Lₙ(X)}. φ is obviously nano semi-open. If A is a non-empty subset of U and A \subseteq Lₙ(X), then N Cl (N Int (A)) = N Cl (φ) = φ. Therefore, A is not nano semi-open, if A \subseteq Lₙ(X). If A \supseteq Lₙ(X), then N Cl (N Int (A)) = N Cl (Lₙ(X)) = U, since Lₙ(X) = Uₙ(X). Therefore, A \subseteq N Cl (N Int (A)) and hence A is nano semi-open. Thus φ and sets containing Lₙ(X) are the only nano semi-open sets in U, if Lₙ(X) = Uₙ(X).

Theorem 4.3 If Lₙ(X) = φ and Uₙ(X) \not\subseteq U, then only those sets containing Uₙ(X) are the nano semi-open sets in U.

Proof: τₙ(X) = {U, φ, Uₙ(X)}. Let A be a non-empty subset of U. If A \subseteq Uₙ(X), then N Cl (N Int (A)) = N Cl (φ) = φ and hence A \not\subseteq N Cl (N Int (A)). Therefore, A is not nano semi-open in U. If A \supseteq Uₙ(X), then N Cl (N Int (A)) = N Cl (Uₙ(X)) = U and hence A \subseteq N Cl (N Int (U)). Therefore, A is nano semi-open in U. Thus, only the sets A such that A \subseteq Uₙ(X) are the only nano semi-open sets in U.

Theorem 4.4 If Uₙ(X) = U is a nano topological space, then U, φ, Lₙ(X) and Bₙ(X) are the only nano semi-open sets in U.

Proof: τₙ(X) = {U, φ, Lₙ(X), Bₙ(X)}. Let A be a non-empty subset of U. If A \subseteq Lₙ(X), then N Cl (N Int (A)) = φ and hence A is not nano semi-open in U. If A = Lₙ(X), then N Cl (N Int (A)) = N Cl (Lₙ(X)) = Lₙ(X) and hence, A \subseteq N Cl (N Int (A)). Therefore, A is nano semi-open in U. If A \supseteq Lₙ(X), then N Cl (N Int (A)) = N Cl (Lₙ(X)) = Lₙ(X). Therefore, A \not\subseteq N Cl (N Int (A)) and hence A is not nano semi-open in U. If A \subseteq Bₙ(X), then N Cl (N Int (A)) = N Cl (Bₙ(X)) = Bₙ(X) and hence A \not\subseteq N Cl (N Int (A)). Therefore, A is nano semi-open in U. If A \supseteq Bₙ(X), then N Cl (N Int (A)) = N Cl (Bₙ(X)) = Bₙ(X) and hence A \not\subseteq N Cl (N Int (A)) and hence A is not nano semi-open in U. Thus, U, φ, Lₙ(X) and Bₙ(X) are the only nano semi-open sets in U, if Uₙ(X) = U and Lₙ(X) \not\subseteq U. Therefore, A \not\subseteq U and Lₙ(X) \not\subseteq U and hence A is nano semi-open in U. Thus, U, φ, Lₙ(X) and Bₙ(X) are the only nano semi-open sets in U, since U.

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and $\phi$ are the only sets in $U$ which are nano-open and nano-closed.

**Theorem 4.5** If $L_g(X) \neq U_g(X)$ where $L_g(X) \neq \phi$ and $U_g(X) \neq U$, then $\bigcup \phi$, $L_g(X)$, $B_g(X)$, $C_g(X)$ are the only nano semi-open sets in $U$.

**Proof:** Let $\tau_g(X) = \{U, \phi, L_g(X), U_g(X), B_g(X)\}$. Let $A$ be a non-empty, proper subset of $U$. If $A \subseteq L_g(X)$, then $\text{Int}(A) = \phi$ and hence, $\text{Cl}(\text{Int}(A)) = \phi$. Therefore, $A$ is not nano semi-open in $U$.

If $A = L_g(X)$, then $\text{Cl}(\text{Int}(A)) = \text{Cl}(L_g(X)) = L_g(X) \cup \{U_g(X)\}$ and hence $A \subseteq \text{Cl}(\text{Int}(A))$. Therefore, $L_g(X)$ is nano semi-open in $U$. If $A \subseteq B_g(X)$, then $\text{Int}(A) = \phi$ and hence $A$ is not nano semi-open in $U$. If $A = B_g(X)$, then $\text{Cl}(\text{Int}(A)) = \text{Cl}(B_g(X)) = B_g(X) \cup \{U_g(X)\}$ and hence $A \subseteq \text{Cl}(\text{Int}(A))$. Therefore, $B_g(X)$ is nano semi-open in $U$. Since $L_g(X)$ and $B_g(X)$ are nano semi-open, $L_g(X) \cup B_g(X) = U_g(X)$ is also nano semi-open in $U$.

Let $A \subseteq U_g(X)$ such that $A$ has at least one element each of $L_g(X)$ and $B_g(X)$. Then $\text{Int}(A) = \phi$ or $L_g(X)$ or $B_g(X)$ and consequently, $\text{Cl}(\text{Int}(A)) = \phi$ or $L_g(X) \cup \{U_g(X)\}$ or $B_g(X) \cup \{U_g(X)\}$ and hence $A \cup \text{Cl}(\text{Int}(A))$. Therefore, $A$ is not nano semi-open in $U$. If $A \supset U_g(X)$, then $\text{Cl}(\text{Int}(A)) = \text{Cl}(U_g(X)) = U$ and hence $A \subseteq \text{Cl}(\text{Int}(A))$. Therefore, $A$ is nano semi-open. If $A$ has a single element each of $L_g(X)$ and $B_g(X)$ and at least one element of $\{U_g(X)\}$, then $\text{Int}(A) = \phi$. Then, $\text{Cl}(\text{Int}(A)) = \phi$ and hence $A$ is not nano semi-open in $U$.

Similarly, when $A$ has a single element of $L_g(X)$ and at least one element of $\{U_g(X)\}$ or a single element of $B_g(X)$ and at least one element of $\{U_g(X)\}$ then $\text{Cl}(\text{Int}(A)) = \phi$ and hence $A$ is not nano semi-open in $U$. When $A = L_g(X) \cup B$ where $B \subseteq \{U_g(X)\}$, then $\text{Cl}(\text{Int}(A)) = \text{Cl}(L_g(X)) = L_g(X) \cup \{U_g(X)\} \supseteq A$. Therefore, $A$ is nano semi-open in $U$. Similarly, if $A = B_g(X) \cup B$ where $B \subseteq \{U_g(X)\}$, then $\text{Cl}(\text{Int}(A)) = B_g(X) \cup \{U_g(X)\} \supseteq A$. Therefore, $A$ is nano semi-open in $U$. Thus, $U$, $\phi$, $L_g(X)$, $U_g(X)$, $B_g(X)$, any set containing $U_g(X)$, $L_g(X) \cup B$ and $B_g(X) \cup B$ where $B \subseteq \{U_g(X)\}$ are the only nano semi-open sets in $U$.

**Theorem 4.6** If $A$ and $B$ are nano semi-open in $U$, then $A \cup B$ is also nano semi-open in $U$. **Proof:** If $A$ and $B$ are nano semi-open in $U$, then $A \subseteq \text{Cl}(\text{Int}(A))$ and $B \subseteq \text{Cl}(\text{Int}(B))$. Consider $A \cup B \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Cl}(\text{Int}(B)) = \text{Cl}(\text{Int}(A) \cup \text{Int}(B)) \subseteq \text{Cl}(\text{Int}(A \cup B))$ and hence $A \cup B$ is nano semi-open.

**Remark 4.7** If $A$ and $B$ are nano semi-open in $U$, then $A \cap B$ is not nano semi-open in $U$. For example, let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a, b\}$. Then $\tau_g(X) = \{U, \phi, \{a\}, \{a, b\}, \{b, c\}\}$. The nano semi-open sets in $U$ are $\phi$, $\{a\}$, $\{a, c\}$, $\{a, b, d\}$, $\{a, b, c, d\}$. If $A = \{a, c\}$ and $B = \{b, c, d\}$, then $A$ and $B$ are nano semi-open but $A \cap B = \{c\}$ is not nano semi-open in $U$.

**Definition 4.8** A subset $A$ of a nano topological space $(U, \tau_g(X))$ is nano-regular open in $U$, if $\text{Int}(\text{Cl}(A)) = A$.

**Example 4.9** Let $U = \{x, y, z\}$ and $U/R = \{\{x\}, \{y, z\}\}$. Let $X = \{x, z\}$. Then the nano topology on $U$ with respect to $X$ is given by $\tau_g(X) = \{U, \phi, \{x\}, \{y, z\}\}$. The nano closed sets are $U$, $\phi$, $\{y, z\}$, $\{x\}$. Also, $\text{Int}(\text{Cl}(A)) = A$ for $A = U, \phi, \{x\}$ and $\{y, z\}$ and hence these sets are nano regular open in $U.$
Theorem 4.10 Any nano regular open set is nano-open.

Proof: If $A$ is nano regular open in $(U, \tau_A(X))$, $A = N\text{Int} (N\text{Cl} (A))$. Then $N\text{Int} (A) = N\text{Int} (N\text{Int} (N\text{Cl} (A))) = N\text{Int} (N\text{Cl} (A)) = A$. That is, $A$ nano-open in $U$.

Remark 4.11 The converse of the above theorem is not true. For example, let $U = \{a, b, c, d, e\}$ with $U/R = \{\{a, b\}, \{c, e\}, \{d\}\}$. Let $X = \{a, d\}$. Then $\tau_A(X) = \{\{a\}, \{a, d\}, \{a, b\}\}$ and the nano closed sets are $U, \phi, \{a, b, c, e\}$. The nano regular open sets are $\phi, \{d\}$ and $\{a, b\}$. Thus, we note that $\{a, b\}$ is nano-open but is not nano regular open. Also, we note that the nano regular open sets do not form a topology, since $\{d\} \cup \{a, b\} = \{a, b, d\}$ is not nano regular open, even though $\{d\}$ and $\{a, b\}$ are nano regular.

Theorem 4.12 In a nano topological space $(U, \tau_A(X))$, if $L_A(X) \neq U_A(X)$, then the only nano regular open sets are $U, \phi, L_A(X)$ and $B_A(X)$.

Proof: The only nano-open sets $(U, \tau_A(X))$ are $U, \phi, L_A(X), U_A(X)$ and $B_A(X)$ and hence the only nano closed sets in $U$ are $U, \phi, \{L_A(X)\}, \{U_A(X)\}$ and $\{B_A(X)\}$ which are respectively $U, \phi, U_A(X)$, $L_A(X)$, and $L_A(X) \cup L_A(X)$.

Case 1: Let $A = L_A(X)$. Then $N\text{Cl} (A) = [B_A(X)]^C \subset \phi \subset U$. Therefore, $N\text{Int} (N\text{Cl} (A)) = N\text{Int} [B_A(X)]^C = \phi \subset U$. Therefore, $A = L_A(X)$ is nano-regular open.

Case 2: Let $A = B_A(X)$. Then $N\text{Cl} (A) = [L_A(X)]^C \subset \phi \subset U$. Therefore, $N\text{Int} (N\text{Cl} (A)) = N\text{Int} [L_A(X)]^C = \phi \subset U$. That is, $A = B_A(X)$ is nano-regular open.

Case 3: If $A = U_A(X)$, then $N\text{Cl} (A) = U$. Therefore, $N\text{Int} (N\text{Cl} (A)) = N\text{Int} (U) = U \neq A$. That is, $A = U_A(X)$ is not nano regular open unless $U_A(X) = U$.

Case 4: Since $N\text{Int} (N\text{Cl} (U)) = U$ and $N\text{Int} (N\text{Cl} (\phi)) = \phi$, and $\phi$ are nano regular open. Also any nano regular open set is nano-open. Thus, $U, \phi, L_A(X)$ and $B_A(X)$ are the only nano regular open sets.

Theorem 4.13: In a nano topological space $(U, \tau_A(X))$, if $L_A(X) = U_A(X)$, then the only nano regular open sets are $U$ and $\phi$.

Proof: The nano-open sets in $U$ are $U, \phi$ and $L_A(X)$ And $N\text{Int} (N\text{Cl} (L_A(X))) = U \neq L_A(X)$. Therefore, $L_A(X)$ is not nano regular open. Thus, the only nano regular open sets are $U$ and $\phi$.

Corollary: If $A$ and $B$ are two nano regular open sets in a nano topological space, Then $A \cap B$ is also nano regular open.

REFERENCES